Inverse Optimal Design of the Distributed Consensus Protocol for Formation Control of Multiple Mobile Robots

Jae Young Lee, Yoon Ho Choi*, and Jin Bae Park

***Abstract*— This paper presents the inverse optimal design method of a nonlinear distributed consensus protocol for formation control of multiple mobile robots. Both dynamics and kinematics are considered in the protocol design. First, we propose a state transformation method to obtain a proper consensus model of a mobile robot. Then, the inverse optimal protocol is designed with respect to a meaningful cost function under the assumption of perfect angular velocity tracking. The assumption will be relaxed by extending the inverse optimal protocol using the backstepping and Lyapunov's direct methods. The numerical simulation is carried out to verify the effectiveness of the proposed method.**

I. Introduction

Formation control of multiple mobile robots have received much attention for many years, and various approaches such as behavioral methods [1], [2], virtual structure [3], [4], leader-follower [5], [6], and graph-theoretic techniques [7], [8] have been developed. Among these approaches for formation control, graph-theoretic technique has the highest degree of freedom of communication among the mobile agents; while the others are designed with a pre-determined fixed (perhaps complex) communication topology, graph-theoretic approaches allow to have arbitrary structure of communication networks that is described by a digraph (directed graph) and satisfies some required properties.

On the other hand, the studies on the formation control of multiple vehicles in graph-theoretic perspectives and the related consensus protocols were mainly done under the assumption that the dynamics of each mobile agent is modeled by a linear system [7]–[14]. Moreover, in the case of optimal consensus algorithms, there is no research for *nonlinear* multi-agent systems to the best authors' knowledge. This is mainly due to the difficulties arising from the constraints on the communication topology of the group of mobile agents. Even for the linear optimal consensus protocols [9]–[11], the optimality of the proposed protocols has not been proven up to date due to those difficulties related to the communication constraints. This problem can be solved by designing the consensus protocols with inverse optimality. In this case, the minimizing performance index is determined *a posteriori* according to the given communication topology (and the designed protocol), so the aforementioned difficulties can be alleviated. The researches on the design of inverse optimal protocol for single integrator dynamics can be found in [12], [13], where Cao and Ren [13] suggest both the control gain and the adjacency matrix of the graph that guarantee the inverse optimality. Furthermore, the recent work [14] done by Movric and Lewis provides the consensus protocol for general linear dynamics, which guarantees the inverse optimality on the directed graph topologies.

The formation consensus protocols of multiple mobile robots are mostly designed based on the linear consensus theory by virtue of dynamic feedback linearization [2], [15] that converts the kinematics of a mobile robot into a simple double integrator. In [16] and [17], the authors proposed nonlinear cooperative formation protocols of mobile robots by employing backstepping and consensus theory for non-holonomic systems [17]. However, most of the formation consensus methods for mobile robots did not consider the dynamics that drives the velocity inputs of the kinematics. Moreover, to the best authors' knowledge, there is no result on the formation consensus protocol for multiple mobile robots that guarantees the nonlinear inverse optimality of the whole closed-loop multi agent system.

In this paper, we propose a distributed nonlinear protocol for formation consensus of multiple mobile robots; the protocol is guaranteed inverse optimal under the simple graph Laplacian and the perfect angular velocity tracking situation, and is designed by considering both dynamics and kinematics of the mobile robots. First, we propose a state transformation technique to derive a proper consensus model from the kinematic and dynamic models of a mobile robot. Based on that and the work in [14], the nonlinear inverse optimal protocol is designed under the assumption of perfect angular velocity tracking, and extended by backstepping control method to relax the assumption and yield the actual driving torque input of the angular velocity. The conditions for Lyapunov's stability and inverse optimality with respect to a meaningful cost function are mathematically shown when the graph Laplacian is simple. Finally, the numerical simulation is performed to verify the effectiveness of the proposed method.

Notations. For real matrices $\mathbf{X} \in \mathbb{R}^{n\times m}$ and $\mathbf{Y} \in \mathbb{R}^{p\times q}$, $\mathbf{X}\otimes\mathbf{Y}$ is the Kronecker product of $\mathbf{X}$ and $\mathbf{Y}$; the null space of a matrix $\mathbf{X}$ is denoted by $\ker(\mathbf{X})$. The $n\times m$ zero matrix and $n\times n$ identity matrix are denoted by $\mathbf{0}_{n\times m}$ and $\mathbf{I}_n$, respectively; $\mathbf{0}_n$ denotes the zero vector in $\mathbb{R}^n$, and $\mathbf{1}_n :=$

This work has been supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2013R1A1A2012851).

J. Y. Lee and J. B. Park are with Department of Electrical and Electronic Engineering, Yonsei University, Yonsei-ro 50, Seodaemum-Gu, Seoul 120-749, Korea (E-mail: jyounglee@yonsei.ac.kr, jbpark@yonsei.ac.kr).

Y. H. Choi is with Department of Electronic Engineering, Kyonggi University, Suwon, Kyonggi-Do 443-760, Korea (E-mail: yhchoi@kyonggi.ac.kr).

* Corresponding author.

$[1, 1, \cdots, 1]^T \in \mathbb{R}^n$ is the vector whose elements are all equal to '1'. For a square matrix $\mathbf{Z} \in \mathbb{R}^{n\times n}$, $\lambda_i(\mathbf{Z})$ denotes the i-th eigenvalue of $\mathbf{Z}$ with $|\lambda_i(\mathbf{Z})| \leq |\lambda_{i+1}(\mathbf{Z})|$ for $i = 1, 2, \cdots, n-1$. $\mathbf{Z} \succ \mathbf{0}_{n\times n}$ (resp. $\mathbf{Z} \succeq \mathbf{0}_{n\times n}$) indicates that $\mathbf{Z}$ is positive definite (resp. positive semi-definite). For any $\mathbf{Z} \succeq \mathbf{0}_{n\times n}$, $\lambda_{\max}(\mathbf{Z})$ is the maximum eigenvalue of $\mathbf{Z}$, *i.e.*, $\lambda_{\max}(\mathbf{Z}) := \lambda_n(\mathbf{Z})$; $\lambda_{<0\,\min}(\mathbf{Z})$ means the minimum among the positive eigenvalues of $\mathbf{Z}$.

II. Graph Theory

In this paper, the communication network among the mobile robots is described by a weighted digraph (directed graph) $\mathcal{G} = \{\mathcal{N}, \mathcal{E}, \mathcal{A}\}$, where $\mathcal{N} := \{1, 2, 3, \cdots, N\}$ is the node set, $\mathcal{E}$ denotes a subset of $\mathcal{N} \times \mathcal{N}$ called the edge set, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the weighted adjacency matrix with its element $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The neighboring set $\mathcal{N}_i$ of a node $i \in \mathcal{N}$ is defined by $\mathcal{N}_i := \{j \in \mathcal{N} : (j, i) \in \mathcal{E}\}$. It is assumed that the digraph $\mathcal{G}$ is simple, *i.e.*,

Assumption 1: $i \notin \mathcal{N}_i$, $\forall i \in \mathcal{N}$.

Define the out-degree matrix $\mathcal{D}$ and the graph Laplacian matrix $\mathbf{L}$ as $\mathcal{D} := \mathrm{diag}\{d_1, d_2, \cdots, d_N\}$ with $d_i = \sum_{j=1}^{N} a_{ij}$ ($i \in \mathcal{N}$) and $\mathbf{L} := \mathcal{D} - \mathcal{A}$, respectively. Note that $\mathbf{L}$ has at least one zero eigenvalue $\lambda_1(\mathbf{L}) = 0$ corresponding to the eigenvector $\mathbf{1}_N$. In this paper, we assume $\mathbf{L}$ is simple, *i.e.*,

Assumption 2: $\lambda_i(\mathbf{L}) \neq \lambda_j(\mathbf{L})$ for all $i, j \in \mathcal{N}$.

A digraph $\mathcal{G}$ is said to be *detailed balanced* if and only if for all $i, j \in \mathcal{N}$, (i) $a_{ij} > 0 \Leftrightarrow a_{ji} > 0$, and (ii) there exist positive constants β_i's such that $\beta_i a_{ij} = \beta_j a_{ji}$.

Proposition 1 ([14]): If the digraph $\mathcal{G}$ satisfies Assumptions 1 and 2, then there exist $\mathbf{S} \succeq \mathbf{0}_{N\times N}$ and $\mathbf{R} \succ \mathbf{0}_{N\times N}$ such that $\mathbf{S} = \mathbf{RL}$. Moreover, if $\mathcal{G}$ is detailed balanced, then $\mathbf{R}$ can be chosen as a diagonal matrix, *e.g.*, $\mathbf{R} = \mathbf{\Pi}$, where $\mathbf{\Pi} := \mathrm{diag}\{\beta_1, \beta_2, \cdots, \beta_N\}$.

III. Dynamic Model of Mobile Robots

The kinematic and dynamic models of each i-th mobile robot shown can be represented as follows [18].

Kinematic:

$$\begin{bmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{\theta}_i \end{bmatrix} = \begin{bmatrix} \nu_i \cos\theta_i \\ \nu_i \sin\theta_i \\ w_i \end{bmatrix} \tag{1}$$

Dynamic:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{11} \end{bmatrix} \begin{bmatrix} \dot{w}_{i,R} \\ \dot{w}_{i,L} \end{bmatrix} + \begin{bmatrix} b & \frac{\alpha w_i}{R} \\ -\frac{\alpha w_i}{R} & b \end{bmatrix} \begin{bmatrix} w_{i,R} \\ w_{i,L} \end{bmatrix} = \begin{bmatrix} \tau_{i,R} \\ \tau_{i,L} \end{bmatrix} \tag{2}$$

where $[x_i, y_i]^T \in \mathbb{R}^2$ and $\theta_i \in \mathbb{R}$ are the position and the angle orientation of the i-th mobile robot; $\nu_i \in \mathbb{R}$ and $w_i \in \mathbb{R}$ are linear and angular velocities of the i-th robot; $w_{i,L} \in \mathbb{R}$ and $w_{i,R} \in \mathbb{R}$ denotes the angular velocities of the left and right wheels of the robot; $\tau_{i,L} \in \mathbb{R}$ and $\tau_{i,R} \in \mathbb{R}$ represents the control torque inputs applied to the robot's left and right wheels, respectively. $b > 0$ is the damping coefficient, $R > 0$ is the half of the width of the mobile robot, and the constant $\alpha > 0$ is defined by $\alpha := r^2 m_c d/2$, where r is the radius of the wheel, d is the distance from the center of mass P_0 of the robot to the middle point between the right and left wheels, and m_c is the mass of the body of the mobile robot. The effective masses m_{11} and m_{12} are given by

$$\begin{cases} m_{11} = I_w + r^2(mR^2 + I)/4R^2 \\ m_{12} = r^2(mR^2 - I)/4R^2 \end{cases}$$
$$(I = m_c d^2 + 2m_w R^2 + I_c + 2I_m,\ m = m_c + 2m_w),$$

where m_w is the mass of a wheel; I_c, I_w, and I_m are the moment of inertia of the body about the vertical axis through P_0, the wheel about the wheel axis and the wheel diameter, respectively.

To derive the useful model, note that the velocities of the mobile robot (ν_i, w_i) and the angular velocities of the wheels $(w_{L,i}, w_{R,i})$ satisfy

$$\begin{cases} \nu_i = r(w_{R,i} + w_{L,i})/2, \\ w_i = r(w_{R,i} - w_{L,i})/2R \end{cases} \tag{3}$$

Differentiating (3) with respect to time and substituting (2) into the resultant equation yields

$$M_1 \dot{\nu}_i = -b\nu_i + \alpha w_i^2 + \tau_{v,i} \tag{4}$$

$$M_2 \dot{w}_i = -bRw_i - \frac{\alpha}{R}\nu_i w_i + \tau_{w,i} \tag{5}$$

where M_1, M_2, $\tau_{v,i}$, and $\tau_{w,i}$ are defined by

$$\begin{cases} M_1 := m_{11} + m_{12}, \quad M_2 := R \cdot (m_{11} - m_{12}), \\ \tau_{v,i} := r(\tau_{R,i} + \tau_{L,i})/2, \quad \tau_{w,i} := r(\tau_{R,i} - \tau_{L,i})/2R. \end{cases}$$

Now, differentiating $\dot{x}$ and $\dot{y}$ in (1) and substituting (4), we have

$$\begin{aligned} M_1 \ddot{x}_i &= M_1 \dot{\nu}_i \cos\theta_i - M_1 \nu_i w_i \sin\theta_i \\ &= \big(-b\nu_i + \alpha w_i^2 + \tau_{v,i}\big) \cdot \cos\theta_i - M_1 \nu_i w_i \sin\theta_i \\ M_1 \ddot{y}_i &= M_1 \dot{\nu}_i \sin\theta_i + M_1 \nu_i w_i \cos\theta_i \\ &= \big(-b\nu_i + \alpha w_i^2 + \tau_{v,i}\big) \cdot \sin\theta_i + M_1 \nu_i w_i \cos\theta_i. \end{aligned}$$

Rearranging these equations, we finally obtain the following nonlinear consensus dynamic model of the mobile robot:

$$\begin{cases} \dot{\mathbf{q}}_i = \mathbf{v}_i \\ \dot{\mathbf{v}}_i = \mathbf{T}(\theta_i, \nu_i)\mathbf{u}_i \end{cases} \tag{6}$$

where $\mathbf{q}_i \in \mathbb{R}^2$ and $\mathbf{v}_i \in \mathbb{R}^2$ are defined as $\mathbf{q}_i := [x_i, y_i]^T$ and $\mathbf{v}_i := \dot{\mathbf{q}}_i$, respectively; $\mathbf{u}_i \in \mathbb{R}^2$ is the effective control input defined by

$$\mathbf{u}_i \equiv \begin{bmatrix} u_{1,i} \\ u_{2,i} \end{bmatrix} := \begin{bmatrix} \tau_{v,i} - b\nu_i + \alpha w_i^2 \\ w_i \end{bmatrix}, \tag{7}$$

and $\mathbf{T}(\theta_i, \nu_i)$ is the transformation matrix given by

$$\mathbf{T}(\theta_i, \nu_i) := \frac{1}{M_1} \begin{bmatrix} \cos\theta_i & -M_1 \nu_i \sin\theta_i \\ \sin\theta_i & M_1 \nu_i \cos\theta_i \end{bmatrix}. \tag{8}$$

For the invertibility of $\mathbf{T}(\theta_i, \nu_i)$, we assume throughout the paper that

Assumption 3: $\nu_i \neq 0$ for all $i \in \mathcal{N}$.

IV. Design of Inverse Optimal Distributed Formation Consensus Protocol

In this section, we design the distributed formation consensus protocol $\mathbf{u}_i$ in (6), which is inverse optimal with respect to the cost function given *a posteriori*. Define $\mathbf{\Gamma}(\nu_i) \succeq \mathbf{0}_{2\times 2}$ for $i \in \mathcal{N}$ as

$$\mathbf{\Gamma}(\nu_i) := \gamma \cdot \mathrm{diag}\{1/M_1^2,\ \nu_i^2\}, \tag{9}$$

where $\gamma > 0$ is a positive constant. We start our discussion with the following two lemmas that dramatically simplify the analysis, and are obvious by the definitions of $\mathbf{T}(\theta_i, \nu_i)$ and $\mathbf{\Gamma}(\nu_i)$ given in (8) and (9), respectively.

Lemma 1: $\forall \nu_i \in \mathbb{R} \setminus \{0\}$, $\mathbf{\Gamma}(\nu_i)$ is positive definite.

Lemma 2: $\forall \nu_i \in \mathbb{R} \setminus \{0\}$ and $\forall \theta_i \in \mathbb{R}$,

$$\mathbf{T}(\theta_i, \nu_i)\mathbf{\Gamma}^{-1}(\nu_i)\mathbf{T}^T(\theta_i, \nu_i) = \gamma^{-1} \cdot \mathbf{I}_2.$$

Now, define the state variable $\mathbf{x}_i \in \mathbb{R}^4$ for i-th robot as

$$\mathbf{x}_i := \begin{bmatrix} \mathbf{q}_i - \mathbf{d}_i \\ \mathbf{v}_i \end{bmatrix}, \tag{10}$$

where $\mathbf{d}_i \in \mathbb{R}^2$ is the distance vector of the i-th robot describing the desired formation. Differentiating (10) and substituting (6), we obtain the following dynamics:

$$\dot{\mathbf{x}}_i = \mathbf{A}\mathbf{x}_i + \mathbf{B}(\theta_i, \nu_i)\mathbf{u}_i, \tag{11}$$

where $\mathbf{A}$ and $\mathbf{B}(\theta_i, \nu_i)$ are defined as

$$\mathbf{A} := \begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{I}_2 \\ \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} \end{bmatrix}, \mathbf{B}(\theta_i, \nu_i) := \begin{bmatrix} \mathbf{0}_{2\times 2} \\ \mathbf{T}(\theta_i, \nu_i) \end{bmatrix}.$$

For the system (11) and a given digraph $\mathcal{G} = \{\mathcal{N}, \mathcal{E}, \mathcal{A}\}$, we propose the following inverse optimal protocol for formation consensus:

$$\mathbf{u}_i = -c\,\mathbf{K}(\theta_i, \nu_i) \sum_{j \in \mathcal{N}_i} a_{ij}(\mathbf{x}_i - \mathbf{x}_j) \tag{12}$$

where $c > 0$ is a constant and $\mathbf{K}(\theta_i, \nu_i)$ is given by

$$\mathbf{K}(\theta_i, \nu_i) := \mathbf{\Gamma}^{-1}(\nu_i)\mathbf{B}^T(\theta_i, \nu_i)\mathbf{P}. \tag{13}$$

$\mathbf{P}$ is the solution of the algebraic Riccati equation (ARE):

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \frac{1}{\gamma}\mathbf{P}\mathbf{B}_0\mathbf{B}_0^T\mathbf{P} + \mathbf{Q} = \mathbf{0}_{4\times 4} \tag{14}$$

for $\mathbf{B}_0 \in \mathbb{R}^{4\times 2}$ defined as $\mathbf{B}_0 := [\, \mathbf{0}_{2\times 2}\ \ \mathbf{I}_2 \,]^T$, and a positive definite matrix $\mathbf{Q} \in \mathbb{R}^{2\times 2}$. Here, the existence of $\mathbf{\Gamma}^{-1}(\nu_i)$ in (13) is guaranteed by Assumption 3 (see Lemma 1).

From (10), we can see that if the formation consensus is reached under Assumption 3, *i.e.*, if $\mathbf{q}_i - \mathbf{q}_j \equiv \mathbf{d}_{ij}$ and $0 \neq \mathbf{v}_i \equiv \mathbf{v}_j$, $\forall i, j \in \mathcal{N}$, where $\mathbf{d}_{ij} := \mathbf{d}_i - \mathbf{d}_j$, then the group of mobile robots moves with the same group velocity, and achieves the desired formation. For notational convenience, we will use the global state vector $\mathbf{x} := [\, \mathbf{x}_1^T\ \ \mathbf{x}_2^T\ \ \cdots\ \ \mathbf{x}_N^T \,]^T \in \mathbb{R}^{4N}$ throughout the paper including the following theorem that states stability and inverse optimality of the protocol (12).

Theorem 1: Consider a digraph $\mathcal{G} = \{\mathcal{N}, \mathcal{E}, \mathcal{A}\}$ satisfying Assumptions 1–2 and the group of mobile robots (11) with the control (12) under Assumption 3. Define the symmetric matrix $\mathbf{\Phi}^T = \mathbf{\Phi} \in \mathbb{R}^{4N\times 4N}$ as

$$\mathbf{\Phi} := \mathbf{S} \otimes \mathbf{Q} + \frac{1}{\gamma}(c \cdot \mathbf{L}^T\mathbf{R}\mathbf{L} - \mathbf{S}) \otimes (\mathbf{P}\mathbf{B}_0\mathbf{B}_0^T\mathbf{P}),$$

where $\mathbf{S}$ and $\mathbf{R}$ are given in Proposition 1 with $\mathbf{S} = \mathbf{R}\mathbf{L}$. Assume that the positive constant $c > 0$ in (12) is sufficiently large so that

$$c > \lambda_{\max}(\mathbf{S})/\lambda_{>0\min}(\mathbf{L}^T\mathbf{R}\mathbf{L}). \tag{15}$$

Then,

1) the formation consensus is reached under (12);
2) $\mathbf{\Phi}$ is positive semi-definite;
3) the protocol (12) minimizes the performance index

$$J(\mathbf{x}(0), \mathbf{u}) = \int_0^\infty \left(c \cdot \mathbf{x}^T\mathbf{\Phi}\,\mathbf{x} + \gamma \cdot \boldsymbol{\mu}^T(\mathbf{R} \otimes \mathbf{I}_2)\boldsymbol{\mu} \right) dt, \tag{16}$$

where $\boldsymbol{\mu} \equiv [\, \boldsymbol{\mu}_1^T\ \ \boldsymbol{\mu}_2^T \cdots \boldsymbol{\mu}_N^T \,]^T \in \mathbb{R}^{2N}$ is the global nonlinear control input in which $\boldsymbol{\mu}_i \in \mathbb{R}^2$ ($i \in \mathcal{N}$) is defined as $\boldsymbol{\mu}_i := \mathbf{T}(\theta_i, \nu_i)\mathbf{u}_i$;
4) the optimal value function $V^*(\mathbf{x}(0)) := \min_{\mathbf{u}} J(\mathbf{x}_0, \mathbf{u})$ is given by

$$V^*(\mathbf{x}(0)) = c \cdot \mathbf{x}^T(0)(\mathbf{S} \otimes \mathbf{P})\mathbf{x}(0). \tag{17}$$

Proof: From Lemma 2 and (13), we have

$$\mathbf{T}(\nu_i, \theta_i)\mathbf{K}(\theta_i, \nu_i) = \frac{1}{\gamma}\mathbf{B}_0^T\mathbf{P},$$
$$\mathbf{B}(\nu_i, \theta_i)\mathbf{K}(\theta_i, \nu_i) = \frac{1}{\gamma}\mathbf{B}_0\mathbf{B}_0^T\mathbf{P}.$$

So, by the definition of $\mathbf{u}_i$ in (12) and $\boldsymbol{\mu}_i = \mathbf{T}(\theta_i, \nu_i)\mathbf{u}_i$, we have

$$\boldsymbol{\mu}_i = -\frac{c}{\gamma} \cdot \mathbf{B}_0^T\mathbf{P} \sum_{j \in \mathcal{N}_i} a_{ij}(\mathbf{x}_i - \mathbf{x}_j), \tag{18}$$

and the system (11) can be expressed in terms of $\boldsymbol{\mu}_i$ as

$$\dot{\mathbf{x}}_i = \mathbf{A}\mathbf{x}_i + \mathbf{B}_0\boldsymbol{\mu}_i. \tag{19}$$

Following the similar procedure to [14, Theorem 2] with (18) and (19), we can see that if $\mathbf{\Psi}$ given by

$$\mathbf{\Psi} := c^2(\mathbf{L} \otimes \frac{1}{\gamma} \cdot \mathbf{B}_0\mathbf{P})^T(\mathbf{R} \otimes \gamma\mathbf{I}_2)(\mathbf{L} \otimes \frac{1}{\gamma} \cdot \mathbf{B}_0\mathbf{P}) - c(\mathbf{R}\mathbf{L} \otimes (\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})) \tag{20}$$

is positive semi-definite, then the formation consensus is reached under the protocol $\boldsymbol{\mu}_i$ (or equivalently, $\mathbf{u}_i$ in (12)) that minimizes the performance index

$$\mathcal{J}(\mathbf{x}_0, \mathbf{u}) = \int_0^\infty \left(\mathbf{x}^T\mathbf{\Psi}\,\mathbf{x} + \gamma \cdot \boldsymbol{\mu}^T(\mathbf{R} \otimes \mathbf{I}_2)\boldsymbol{\mu} \right) dt,$$

and the optimal value function $V^*(\mathbf{x}(0))$ is given by (17). Using the properties of Kronecker products and substituting the ARE (14) and $\mathbf{S}=\mathbf{RL}$, we can prove the equivalence "$\boldsymbol{\Psi}=c\boldsymbol{\Phi}$" as shown below:

$$\begin{aligned}\frac{\boldsymbol{\Psi}}{c} &= c\left(\mathbf{L}^T\mathbf{RL}\otimes\frac{\mathbf{PB}_0^T\mathbf{B}_0\mathbf{P}}{\gamma}\right)-\left(\mathbf{RL}\otimes(\mathbf{A}^T\mathbf{P}+\mathbf{PA})\right)\\ &= c\left[\mathbf{L}^T\mathbf{RL}\otimes\frac{\mathbf{PB}_0^T\mathbf{B}_0\mathbf{P}}{\gamma}\right]+\left[\mathbf{S}\otimes\left(\mathbf{Q}-\frac{\mathbf{PB}_0\mathbf{B}_0^T\mathbf{P}}{\gamma}\right)\right]\\ &= \frac{1}{\gamma}(c\cdot\mathbf{L}^T\mathbf{RL}-\mathbf{S})\otimes\mathbf{PB}_0\mathbf{B}_0^T\mathbf{P})+\mathbf{S}\otimes\mathbf{Q}=\boldsymbol{\Phi}.\end{aligned}$$

This implies the equivalence of the two performance indices $\mathcal{J}(\mathbf{x}_0,\mathbf{u})$ and $J(\mathbf{x}_0,\mathbf{u})$. So, it remains to show that $\boldsymbol{\Phi}$ $(=\boldsymbol{\Psi}/c)$ is positive semi-definite under (15).

To complete the proof, we now prove

$$c\mathbf{S}^T\mathbf{R}^{-1}\mathbf{S}-\mathbf{S}\succeq\mathbf{0}_{N\times N} \tag{21}$$

under (15), which obviously implies $\boldsymbol{\Phi}\succeq\mathbf{0}_{N\times N}$ (see the definition of $\boldsymbol{\Phi}$ and notice that $\mathbf{S}^T\mathbf{R}^{-1}\mathbf{S}=\mathbf{L}^T\mathbf{RL}$).

1) *Proof of* "$\ker(\mathbf{S})=\ker\left(\mathbf{S}^T\mathbf{R}^{-1}\mathbf{S}\right)$": Assumption 2 implies that the zero eigenvalue "0" of the graph Laplacian matrix $\mathbf{L}$ is simple. From this and $\mathbf{S}=\mathbf{RL}$, we have

$$\operatorname{rank}(\mathbf{S})=N-1. \tag{22}$$

In addition, (22) and $\operatorname{rank}(\mathbf{R})=N$ imply

$$\operatorname{rank}\left(\mathbf{S}^T\mathbf{R}^{-1}\mathbf{S}\right)=N-1. \tag{23}$$

Next, for a nonzero vector $\mathbf{z}\in\mathbb{R}^N$, suppose $\mathbf{z}\in\ker(\mathbf{S})$ so $\mathbf{Sz}=\mathbf{0}_N$ holds. Then, we have $\mathbf{S}^T\mathbf{R}^{-1}\mathbf{Sz}=\mathbf{0}_N$, and thereby $\ker(\mathbf{S})\subseteq\ker\left(\mathbf{S}^T\mathbf{R}^{-1}\mathbf{S}\right)$. This yields $\ker(\mathbf{S})=\ker\left(\mathbf{S}^T\mathbf{R}^{-1}\mathbf{S}\right)$ since (22) and (23) imply $\operatorname{nullity}(\mathbf{S})=\operatorname{nullity}\left(\mathbf{S}^T\mathbf{R}^{-1}\mathbf{S}\right)=1$.

2) *Proof of* (21): Since $\mathbf{S}$ and $\mathbf{S}^T\mathbf{R}^{-1}\mathbf{S}$ are positive semi-definite, the zero eigenvalue is the minimum among the eigenvalues of the matrices. Therefore, for $\mathbf{y}\in\mathbb{R}^N$ satisfying $\mathbf{Sy}\neq\mathbf{0}_N$, the inequality

$$\begin{aligned}c\cdot\mathbf{y}^T\mathbf{S}^T\mathbf{R}^{-1}\mathbf{Sy} &\geq c\cdot\lambda_{>0\min}(\mathbf{S}^T\mathbf{R}^{-1}\mathbf{S})\cdot\|\mathbf{y}\|^2\\ &> \lambda_{\max}(\mathbf{S})\cdot\|\mathbf{y}\|^2\geq\mathbf{y}^T\mathbf{Sy}>0\end{aligned}$$

holds by (15); from this and $\ker(\mathbf{S})=\ker\left(\mathbf{S}^T\mathbf{R}^{-1}\mathbf{S}\right)$, we conclude that (21) is satisfied whenever (15) holds. ■

If the digraph $\mathcal{G}$ is detailed balanced, then the performance index (16) in Theorem 1 can be rewritten in a more tractable form as shown in the next theorem.

Theorem 2: Suppose the digraph $\mathcal{G}$ is detailed balanced. Then, under the same conditions of Theorem 1, (16) can be expressed as

$$J(\mathbf{x}(0),\mathbf{u})=\int_0^\infty\left(c\cdot\mathbf{x}^T\boldsymbol{\Phi}\,\mathbf{x}+\sum_{i=1}^N\beta_i\mathbf{u}_i^T\boldsymbol{\Gamma}(\theta_i,\nu_i)\mathbf{u}_i\right)dt, \tag{24}$$

where $\beta_i>0$ $(i\in\mathcal{N})$ is defined in Proposition 1 and satisfies $\beta_ia_{ij}=\beta_ja_{ji}$ for all $i,j\in\mathcal{N}$.

Proof: Since the digraph $\mathcal{G}$ is detailed balanced, we can choose $\mathbf{R}$ in Theorem 1 as $\mathbf{R}=\operatorname{diag}\{\beta_1,\beta_2,\cdots,\beta_N\}$ by Proposition 1. This implies

$$\mathbf{R}\otimes\mathbf{I}_2=\operatorname{blockdiag}\{\beta_1\mathbf{I}_2,\beta_2\mathbf{I}_2,\cdots,\beta_N\mathbf{I}_2\},$$

so the input-term "$\gamma\cdot\boldsymbol{\mu}^T\left(\mathbf{R}\otimes\mathbf{I}_2\right)\boldsymbol{\mu}$" in (16) becomes

$$\gamma\cdot\boldsymbol{\mu}^T\left(\mathbf{R}\otimes\mathbf{I}_2\right)\boldsymbol{\mu}=\gamma\cdot\sum_{i=1}^N\beta_i\boldsymbol{\mu}_i^T\boldsymbol{\mu}_i=\sum_{i=1}^N\beta_i\mathbf{u}_i^T\boldsymbol{\Gamma}(\theta_i,\nu_i)\mathbf{u}_i,$$

where we have substituted the equalities $\boldsymbol{\mu}_i=\mathbf{T}(\theta_i,\nu_i)\mathbf{u}_i$ and $\gamma\mathbf{T}^T(\theta_i,\nu_i)\mathbf{T}(\theta_i,\nu_i)=\boldsymbol{\Gamma}(\theta_i,\nu_i)$ (see Lemma 2). So, we obtain (24) from (16), which completes the proof. ■

V. Extension by Backstepping Design

To determine the actual torque inputs $\tau_{i,R}$ and $\tau_{i,L}$ of each mobile robot (1)–(2), we should derive each angular torque control $\tau_{i,w}$ in (5) as well as the linear torque control $\tau_{i,v}$ already given in the previous section. In this section, we use the backstepping technique with the w_i-dynamics (5) and the inverse optimal control scheme shown in Section IV to derive each $\tau_{i,w}$. First, note that the dynamics (11) of a mobile robot can be rewritten as

$$\dot{\mathbf{x}}_i=\mathbf{A}\mathbf{x}_i+\mathbf{B}(\theta_i,\nu_i)\left[\mathbf{u}_{i,d}+\tilde{\mathbf{u}}_i\right], \tag{25}$$

where $\mathbf{u}_{i,d}$ is the target control input to the system (25) and defined as $\mathbf{u}_{i,d}:=[\,\tau_{v,i}-b\nu_i+\alpha w_i^2,\ w_{i,d}\,]^T$ for the desired angular velocity $w_{i,d}$; $\tilde{\mathbf{u}}_i$ is defined by $\tilde{\mathbf{u}}_i:=[\,0,\ \tilde{w}_i\,]^T$ with the angular velocity error $\tilde{w}_i:=w_i-w_{i,d}$. We also denote $\tilde{\mathbf{w}}:=[\,\tilde{w}_1^T\ \tilde{w}_2^T\cdots\tilde{w}_N^T\,]^T\in\mathbb{R}^N$ for notational convenience.

The objective of the backstepping design is to derive the angular torque control $\tau_{i,w}$ that drives w_i to asymptotically make the angular velocity error $\tilde{w}_i$ zero and achieve the formation consensus when the target input $\mathbf{u}_{i,d}$ is given by

$$\mathbf{u}_{i,d}=-c\,\mathbf{K}(\theta_i,\nu_i)\sum_{j\in\mathcal{N}_i}a_{ij}(\mathbf{x}_i-\mathbf{x}_j), \tag{26}$$

according to the inverse optimal protocol (12). Now, let $\boldsymbol{\mu}_d,\tilde{\boldsymbol{\mu}}\in\mathbb{R}^{2N}$ be defined as $\boldsymbol{\mu}_d:=[\,\boldsymbol{\mu}_{1,d}^T\ \boldsymbol{\mu}_{2,d}^T\cdots\boldsymbol{\mu}_{N,d}^T\,]^T$ and $\tilde{\boldsymbol{\mu}}:=[\,\tilde{\boldsymbol{\mu}}_1^T\ \tilde{\boldsymbol{\mu}}_2^T\cdots\tilde{\boldsymbol{\mu}}_N^T\,]^T$ with $\boldsymbol{\mu}_{i,d},\tilde{\boldsymbol{\mu}}_i\in\mathbb{R}^2$ $(i\in\mathcal{N})$ given by $\boldsymbol{\mu}_{i,d}:=\mathbf{T}(\theta_i,\nu_i)\mathbf{u}_{i,d}$ and $\tilde{\boldsymbol{\mu}}_i:=\mathbf{T}(\theta_i,\nu_i)\tilde{\mathbf{u}}_i$, respectively. Then, (25) and (26) can be rewritten as

$$\dot{\mathbf{x}}_i=\mathbf{A}\mathbf{x}_i+\mathbf{B}_0\left[\boldsymbol{\mu}_{i,d}+\tilde{\boldsymbol{\mu}}_i\right], \tag{27}$$

$$\boldsymbol{\mu}_{i,d}=-\frac{c}{\gamma}\cdot\mathbf{B}_0^T\mathbf{P}\sum_{j\in\mathcal{N}_i}a_{ij}(\mathbf{x}_i-\mathbf{x}_j)=-\frac{c}{\gamma}\sum_{j=1}^N l_{ij}\,\mathbf{B}_0^T\mathbf{P}\mathbf{x}_i.$$

Combining the control inputs $\boldsymbol{\mu}_{i,d}$ and the dynamic equations (27) $\forall i\in\mathcal{N}$, we obtain

$$\begin{aligned}\boldsymbol{\mu}_d &= -\frac{c}{\gamma}\cdot(\mathbf{L}\otimes\mathbf{B}_0^T\mathbf{P})\mathbf{x},\\ \dot{\mathbf{x}} &= (\mathbf{I}_N\otimes\mathbf{A})\mathbf{x}+(\mathbf{I}_N\otimes\mathbf{B}_0)\left[\boldsymbol{\mu}_d+\tilde{\boldsymbol{\mu}}\right]\\ &\equiv \bar{\mathbf{A}}\mathbf{x}+(\mathbf{I}_N\otimes\mathbf{B}_0)\tilde{\boldsymbol{\mu}},\end{aligned} \tag{28}$$

where $\bar{\mathbf{A}}:=(\mathbf{I}_N\otimes\mathbf{A})-c\,(\mathbf{L}\otimes\mathbf{B}_0\mathbf{B}_0^T\mathbf{P})/\gamma$. For the design and analysis, we essentially need the following lemma, which can be easily proven by considering the inverse optimality given in Theorem 1 (see [14] for more discussions).

Lemma 3: Under the same conditions of Theorem 1, the following global ARE holds:

$$\bar{\mathbf{A}}^T(\mathbf{S}\otimes\mathbf{P})+(\mathbf{S}\otimes\mathbf{P})\bar{\mathbf{A}} = -\mathbf{\Phi}-\frac{c}{\gamma}\cdot(\mathbf{S}\otimes\mathbf{P})(\mathbf{I}_N\otimes\mathbf{B}_0)(\mathbf{I}_N\otimes\mathbf{B}_0)^T(\mathbf{S}\otimes\mathbf{P}).$$

Theorem 3: Consider a group of mobile robots described by (25) for each $i\in\mathcal{N}$ and a digraph $\mathcal{G}=\{\mathcal{N},\mathcal{E},\mathcal{A}\}$. Let each $\tau_{w,i}$ be given by

$$\tau_{w,i}=\big(bR+\frac{\alpha}{R}\,\nu_i\big)w_i+M_2\,\dot{w}_{i,d}-(k_{1,i}+k_{2,i}\nu_i^2)\tilde{w}_i \quad (29)$$

for some $k_{1,i},k_{2,i}>0$. Then, under the same conditions of Theorem 1, the virtual and actual control inputs (26) and (29) asymptotically stabilize the closed-loop system consisting of all of the i-th subsystems (5) and (25) to the formation consensus error space Ω defined as

$$\Omega:=\{(\mathbf{x},\tilde{\mathbf{w}})\in\mathbb{R}^{4N+N}:\mathbf{x}\in\ker(\mathbf{L}),w_i=w_{i,d}\}.$$

Proof: Consider $V^*(\mathbf{x})=c\cdot\mathbf{x}^T(\mathbf{S}\otimes\mathbf{P})\mathbf{x}$ given in (17). Letting $\mathbf{z}:=c(\mathbf{I}_N\otimes\mathbf{B}_0)^T(\mathbf{S}\otimes\mathbf{P})\mathbf{x}$, differentiating $V^*(\mathbf{x})$ with respect to the system (28), and using Lemma 3, we obtain

$$\begin{aligned}\dot{V}^*(\mathbf{x})&=c\cdot\mathbf{x}^T\Big[\bar{\mathbf{A}}^T(\mathbf{S}\otimes\mathbf{P})+(\mathbf{S}\otimes\mathbf{P})\bar{\mathbf{A}}\Big]\mathbf{x}+2\mathbf{z}^T\tilde{\boldsymbol{\mu}}\\&=-c\cdot\mathbf{x}^T\mathbf{\Phi}\mathbf{x}-\frac{1}{\gamma}\cdot\|\mathbf{z}\|^2+2\mathbf{z}^T\tilde{\boldsymbol{\mu}}.\end{aligned}$$

Applying Young's inequality $2\mathbf{z}^T\tilde{\boldsymbol{\mu}}\le\gamma^{-1}\cdot\|\mathbf{z}\|^2+\gamma\cdot\|\tilde{\boldsymbol{\mu}}\|^2$, using Lemma 2 with the substitution of $\tilde{\boldsymbol{\mu}}_i=\mathbf{T}(\theta_i,\nu_i)\tilde{\mathbf{u}}_i$, and rearranging the result with $\tilde{\mathbf{u}}_i=[\,0,\ \tilde{w}_i\,]^T$ and (9) yields

$$\begin{aligned}\dot{V}^*(\mathbf{x})&\le-c\,\mathbf{x}^T\mathbf{\Phi}\mathbf{x}+\sum_{i=1}^N\tilde{\mathbf{u}}_i^T\underbrace{\Big(\gamma\mathbf{T}^T(\theta_i,\nu_i)\mathbf{T}(\theta_i,\nu_i)\Big)}_{=\mathbf{\Gamma}(\nu_i)\text{by Lemma 2}}\tilde{\mathbf{u}}_i\\&=-c\,\mathbf{x}^T\mathbf{\Phi}\mathbf{x}+\gamma\cdot\sum_{i=1}^N\nu_i^2\tilde{w}_i^2.\end{aligned}\quad (30)$$

Now, consider $\bar{V}(\mathbf{x},\tilde{\mathbf{w}})=V^*(\mathbf{x})+\frac{\gamma}{2\varepsilon}\cdot\sum_{i=1}^N M_2\tilde{w}_i^2$ with $\varepsilon\in(0,1]$ as a Lyapunov function candidate. Differentiating $\bar{V}(\mathbf{x},\tilde{\mathbf{w}})$ along the trajectory generated by the systems (5) and (28) and substituting (30), we obtain

$$\begin{aligned}\dot{\bar{V}}(\mathbf{x},\tilde{\mathbf{w}})&\le-c\,\mathbf{x}^T\mathbf{\Phi}\mathbf{x}+\gamma\cdot\sum_{i=1}^N\Big(\nu_i^2\tilde{w}_i^2+\frac{M_2}{\varepsilon}\cdot\tilde{w}_i\dot{\tilde{w}}_i\Big)\\&=-c\,\mathbf{x}^T\mathbf{\Phi}\mathbf{x}+\gamma\cdot\sum_{i=1}^N\tilde{w}_i\cdot\Big[\nu_i^2\tilde{w}_i\\&\quad+\frac{1}{\varepsilon}\cdot\Big(-bRw_i-\frac{\alpha}{R}\nu_iw_i+\tau_{w,i}-M_2\dot{w}_{i,d}\Big)\Big]\end{aligned}\quad (31)$$

Define $\kappa_{1,i},\kappa_{2,i}>0$ as $\kappa_{1,i}:=k_{1,i}/\varepsilon$ and $\kappa_{2,i}:=k_{2,i}/\varepsilon$. Then, substituting (29) into (31) yields

$$\dot{\bar{V}}(\mathbf{x},\tilde{\mathbf{w}})\le-c\,\mathbf{x}^T\mathbf{\Phi}\mathbf{x}-\gamma\cdot\sum_{i=1}^N\Big(\kappa_{1,i}+(\kappa_{2,i}-1)\nu_i^2\Big)\tilde{w}_i^2.$$

Therefore, $\dot{\bar{V}}$ is negative definite if $\kappa_{1,i}>0$ and $\kappa_{2,i}\ge1$, which become $k_{1,i}>0$ and $k_{2,i}\ge\varepsilon>0$ by the definition of $\kappa_{1,i}$ and $\kappa_{2,i}$. Since ε can be any value in $(0,1]$, the latter condition implies $k_{2,i}>0$, so the proof is completed by Lyapunov's theorem for partial stability [14]. ■

VI. Simulation Results

To verify the performance of the proposed protocol (26) and (29) designed with the inverse optimal consensus and the backstepping control methodologies, respectively, we carried out a numerical simulation with the three mobile robots ($\mathcal{N}=\{1,2,3\}$) whose kinematic and dynamic models are given by (1) and (2), respectively. In the simulation, the parameters of the mobile robots are given by $R=0.75$ [m], $d=0.3$ [m], $r=0.15$ [m], $m_c=30$ [kg], $m_w=1$ [kg], $I_c=15.625$ [kg·m^2], $I_w=0.005$ [kg·m^2], $I_m=0.0025$ [kg·m^2], and $b=2$ [m]; we considered a detailed balanced graph $\mathcal{G}$ with its graph Laplacian matrix $\mathbf{L}$ given by

$$\mathbf{L}=\begin{bmatrix}0.5&-0.5&0.0\\-0.4&0.6&-0.2\\0.0&-2.0&2.0\end{bmatrix},$$

which obviously satisfies $\mathbf{S}=\mathbf{R}\mathbf{L}$ in Proposition 1 with $\mathbf{R}=\mathrm{diag}\{1,\frac{5}{2},\frac{1}{4}\}$, and gives the inverse optimality condition $c>2.2978$ by (15). From this, we choose $c=3$. The matrix $\mathbf{Q}$ and the positive constant γ in the ARE (14) are set to $\mathbf{Q}=\mathbf{I}_4$ and $\gamma=1$. The positive gains $k_{i,1}$ and $k_{i,2}$ in (29) are given by $k_{i,1}=k_{i,2}=5$ for all $i\in\{1,2,3\}$. Initial positions $\boldsymbol{p}_i^0=(x_i(0),y_i(0),\theta_i(0))$, initial velocities $\boldsymbol{\xi}_i^0=(\nu_i(0),w_i(0))$, and the distance vectors $\mathbf{d}_i$ of the mobile robots are

$$\begin{aligned}&\boldsymbol{p}_1^0=(0,0,\pi/6),\quad&&\boldsymbol{\xi}_1^0=(2,0),\quad&&\mathbf{d}_1=[\,3.0,\quad 0.0\,]^T,\\&\boldsymbol{p}_2^0=(0,1,0),&&\boldsymbol{\xi}_2^0=(1,0),&&\mathbf{d}_2=[\,0.0,\quad 0.3\,]^T,\\&\boldsymbol{p}_3^0=(1,0,-\pi/6),&&\boldsymbol{\xi}_3^0=(3,0),&&\mathbf{d}_3=[\,0.0,\ -0.3\,]^T.\end{aligned}$$

The simulation results are shown in Figs. 1 and 2, where Fig. 1 describes the position trajectories of mobile robots in xy-plane, and Fig. 2 illustrates the variations of mobile robots' linear and angular velocities (ν_i,w_i). As shown in Figs. 1 and 2(a), the mobile robots driven by the proposed protocol (26) and (29) shape and maintain the desired formation, and ultimately move with the same consensus velocities and angle orientations. In case of the angular velocity command w_i, it converges to zero as shown in Fig. 2(b). This is because the desired angular velocity $w_{i,d}$ becomes zero when the formation consensus is reached so that $\mathbf{x}\in\ker(\mathbf{L}\otimes\mathbf{I}_4)$, and $\tau_{i,w}$ is designed to achieve $w_i\to w_{i,d}$ as $t\to\infty$; So, after the consensus is reached, there is no change in the angle orientation θ_i of each mobile robot (see Fig. 1).

VII. Conclusions

In this paper, we presented a state transformation technique by extending the dynamic feedback linearization to obtain a proper consensus model from the kinematics and dynamics of a mobile robot. Then, using the properties of the transformed model, inverse optimal consensus theory,

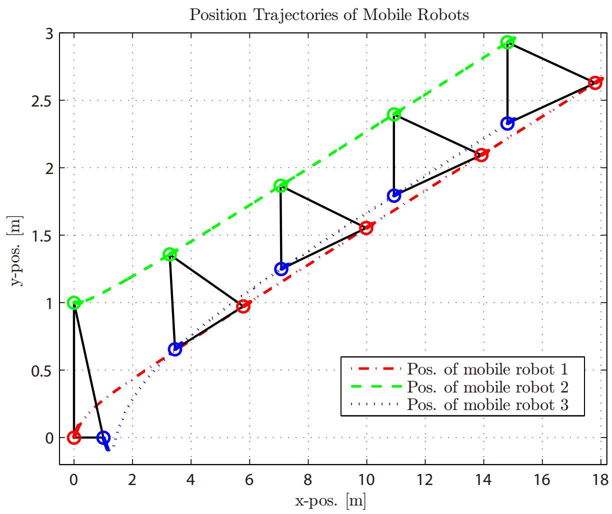
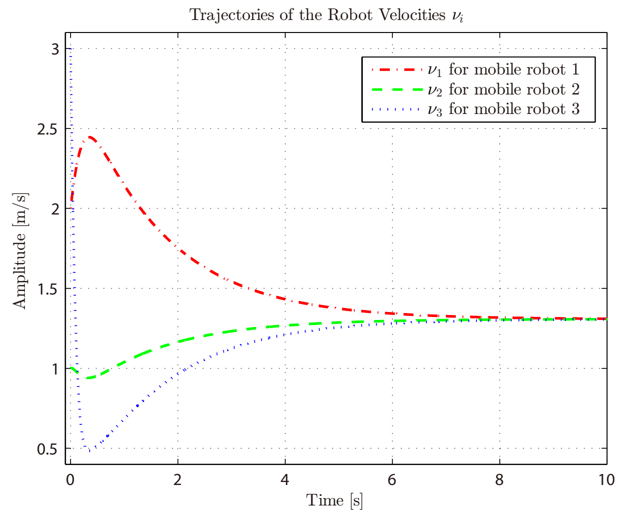
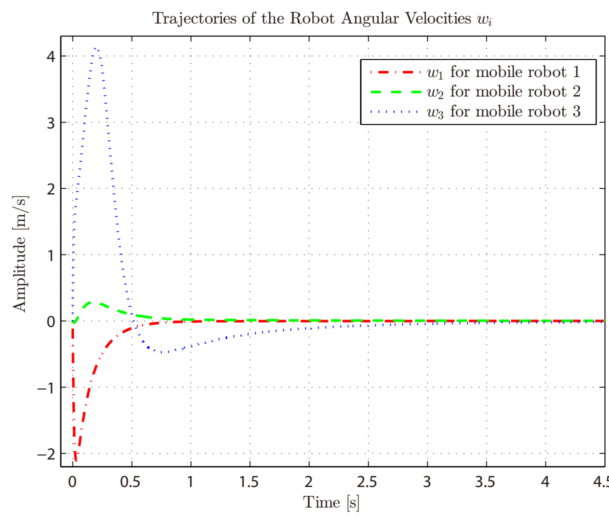


Fig. 1. Position trajectories of mobile robots in xy-plane.



(a) Trajectories of linear velocities ν_i



(b) Trajectories of angular velocities w_i

Fig. 2. Velocity commands (ν_i, w_i) of mobile robots generated by the protocols (26) and (29)—(a) linear velocities ν_i, (b) angular velocities w_i

and backstepping, a nonlinear distributed protocol was designed for formation consensus of multiple mobile robots. By mathematical analysis, we provided the conditions for Lyapunov's cooperative stability and inverse optimality with respect to a meaningful cost function, *e.g.*, the required consensus gain inequality (15), under the directed graph communication topology with a simple graph Laplacian. The numerical simulation results supports the effectiveness of the proposed method.

References

[1] T. Balch and R. C. Arkin, "Behavior-based formation control for multirobot teams," *IEEE Trans. Robotics and Automation*, vol. 14, no. 6, pp. 926–939, 1998.

[2] J. R. T. Lawton, R. W. Beard, and B. J. Young, "A decentralized approach to formation maneuvers," *IEEE Trans. Robotics and Automation*,, vol. 19, no. 6, pp. 933–941, 2003.

[3] M. A. Lewis and K.-H. Tan, "High precision formation control of mobile robots using virtual structures," *Autonomous Robots*, vol. 4, no. 4, pp. 387–403, 1997.

[4] K. D. Do, "Formation tracking control of unicycle-type mobile robots with limited sensing ranges," *IEEE Trans. Control Systems Technology*, vol. 16, no. 3, pp. 527–538, 2008.

[5] J. P. Desai, J. P. Ostrowski, and V. Kumar, "Modeling and control of formations of nonholonomic mobile robots," *IEEE Trans. Robotics and Automation*, vol. 17, no. 6, pp. 905–908, 2001.

[6] B. S. Park, J. B. Park, and Y. H. Choi, "Adaptive formation control of electrically driven nonholonomic mobile robots with limited information," *IEEE Trans. Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 41, no. 4, pp. 1061–1075, 2011.

[7] W. Ren *et al.*, "Information consensus in multivehicle cooperative control," *IEEE Control systems magazine*, vol. 27, no. 2, pp. 71–82, 2007.

[8] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1465–1476, 2004.

[9] F. Borrelli and T. Keviczky, "Distributed lqr design for identical dynamically decoupled systems," *Autom. Control, IEEE Transactions on*, vol. 53, no. 8, pp. 1901–1912, 2008.

[10] H. Zhang, F. L. Lewis, and A. Das, "Optimal design for synchronization of cooperative systems: state feedback, observer and output feedback," *IEEE Trans. Autom. Control*, vol. 56, no. 8, pp. 1948–1952, 2011.

[11] W. Dong, "Distributed optimal control of multiple systems," *Int. Journal of Control*, vol. 83, no. 10, pp. 2067–2079, 2010.

[12] Z. Qu and M. Simaan, "Inverse optimality of cooperative control for networked systems," in *In Proc. 48th IEEE Conf. Decision and Control held jointly with the 28th Chinese Control Conference. CDC/CCC 2009*, pp. 1651–1658, 2009.

[13] Y. Cao and W. Ren, "Optimal linear-consensus algorithms: an lqr perspective," *IEEE Trans. Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 40, no. 3, pp. 819–830, 2010.

[14] K. H. Movric and F. L. Lewis, "Cooperative optimal control for multi-agent systems on directed graph topologies," *IEEE Trans. Autom. Control*, vol. 59, no. 3, pp. 769–774, 2014.

[15] G. Oriolo, A. De Luca, and M. Vendittelli, "WMR control via dynamic feedback linearization: design, implementation, and experimental validation," *IEEE Trans. Control Systems Technology*, vol. 10, no. 6, pp. 835–852, 2002.

[16] W. Dong and J. A. Farrell, "Decentralized cooperative control of multiple nonholonomic dynamic systems with uncertainty," *Automatica*, vol. 45, no. 3, pp. 706–710, 2009.

[17] W. Dong, "Flocking of multiple mobile robots based on backstepping," *IEEE Trans. Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 41, no. 2, pp. 414–424, 2011.

[18] B. S. Park, S. J. Yoo, J. B. Park, and Y. H. Choi, "A simple adaptive control approach for trajectory tracking of electrically driven nonholonomic mobile robots," *IEEE Trans. Control Systems Technology*, vol. 18, no. 5, pp. 1199–1206, 2010.