Abstract σ-Algebra and Measurable Functions

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This note summarizes some of the basis theory on σ-algebra and measurable functions, which have served as the fundamentals of measure theory, Lebesque integration, probability, etc. The materials either directly come from or strongly related to those in Chapters 1.2, 2.1, and 2.4 in Folland's real analysis book. Related to them, this note also introduces some basic theory of topology and set theory that also make the statements about σ-algebra and measurable functions clear and rigorously true.

I. Preliminaries

In this note, $\begin{cases} X \text{ and } Y \text{ denote any non-empty sets;} \\ f : X \to Y \text{ is any mapping from } X \text{ to } Y; \\ \mathcal{M} \subseteq \mathcal{P}(X) \text{ and } \mathcal{N} \subseteq \mathcal{P}(Y) \text{ denote any (or some given) } \sigma\text{-algebras on } X \text{ and } Y \text{, respectively.} \end{cases}$

$(X, \mathcal{M})$ (and $(Y, \mathcal{N})$) is called a measurable space. Here, a σ-algebra $\mathcal{M}$ on X (or $\mathcal{N}$ on Y) is precisely defined as follows.

Definition 1. *A non-empty subset $\mathcal{M} \subseteq \mathcal{P}(X)$ is said to be a σ-algebra on X iif:*

$$1)\ \{E_j\}_{j\in\mathbb{N}} \subseteq \mathcal{M} \implies \bigcup_{j\in\mathbb{N}} E_j \in \mathcal{M} \qquad \textit{(closed under countable unions),}$$

$$2)\ E \in \mathcal{M} \implies E^c \in \mathcal{M} \qquad \textit{(closed under complements).}$$

If the first property is replaced by

$$1')\ \{E_j\}_{j=0}^{N} \subseteq \mathcal{M} \implies \bigcup_{j=1}^{N} E_j \in \mathcal{M} \qquad \textit{(closed under finite unions),}$$

then, $\mathcal{M}$ is said to be an algebra on X.

Note that any σ-algebra on X is an algebra on X, but not vice versa. The basic properties of a σ-algebra are as follows.

Proposition 1. *The followings hold for any σ-algebra $\mathcal{M}$ on X and*

1) ϕ, $X \in \mathcal{M}$,
2) $\{E_j\}_{j\in\mathbb{N}} \subseteq \mathcal{M} \implies \bigcap_{j\in\mathbb{N}} E_j \in \mathcal{M}$ *(closed under countable intersections).*
3) $E,\ F \in \mathcal{M} \implies E \setminus F \in \mathcal{M}$ *and* $E \triangle F \in \mathcal{M}$.

Proof. The second part can be directly proven as follows:

$$\bigcap_{j\in\mathbb{N}} E_j = \Big(\bigcup_{j\in\mathbb{N}} E_j^c\Big)^c \in \mathcal{M}.$$

Then, the first part is obvious since for $E \in \mathcal{M}$, we have $X = E \cup E^c \in \mathcal{M}$ and $\phi = E \cap E^c \in \mathcal{M}$. Likewise, since $\mathcal{M}$ is closed under countable unions, countable intersections, and complements, we finally have $E \setminus F = E \cap F^c \in \mathcal{M}$ and $E \triangle F = (E \setminus F) \cup (F \setminus E) \in \mathcal{M}$. $\square$

Example 1. *$\mathcal{M} = \mathcal{P}(X)$ and $\mathcal{M} = \{X, \phi\}$ are the largest and smallest σ-algebra on X, respectively.*

The following shows that the (uncountable) intersection of σ-algebras is also a σ-algebra.

Proposition 2. *Let $\{\mathcal{M}_\alpha\}_{\alpha\in A}$ be a family of σ-algebras on X. Then, $\mathcal{M} = \bigcap_{\alpha\in A} \mathcal{M}_\alpha$ is a σ-algebra.*

Proof. **(Closed under complements)** Assume $E \in \mathcal{M}$. Then, by definition, $E \in \mathcal{M}_\alpha$ for all $\alpha \in A$, which again implies $E^c \in \mathcal{M}_\alpha$ for all $\alpha \in A$. Hence, we obtain $E^c \in \bigcap_{\alpha\in A} \mathcal{M}_\alpha = \mathcal{M}$.
(Closed under countable unions) Assume $\{E_j\}_{j\in\mathbb{N}} \subseteq \mathcal{M}$. Then, $E_j \in \mathcal{M}$ implies that $E_j \in \mathcal{M}_\alpha$ for all $\alpha \in A$. Since $\mathcal{M}_\alpha$ is a σ-algebra, we obtain $\bigcup_{j\in\mathbb{N}} E_j \in \mathcal{M}_\alpha$ for all $\alpha \in A$, implying that $\bigcup_{j\in\mathbb{N}} E_j \in \mathcal{M}$. $\square$

Definition 2. *For any family $\mathcal{E} \subseteq \mathcal{P}(X)$, $\sigma(\mathcal{E})$ denotes the smallest σ-algebra on X that contains $\mathcal{E}$; we call $\sigma\{\mathcal{E}\}$ the σ-algebra generated by $\mathcal{E}$.*

Remark 1. *Note that for any family $\mathcal{E} \subseteq \mathcal{P}(X)$, there is at least one σ-algebra, namely, the power set $\mathcal{P}(X)$ itself, the largest σ-algebra, that contains $\mathcal{E}$. Moreover, since any (uncountable) intersection of σ-algebras is also a σ-algebra as shown in Proposition 2, the smallest σ-algebra $\sigma(\mathcal{E})$ in Definition 2 can be constructed and recognized as the intersection of all σ-algebras containing $\mathcal{E}$.*

By Proposition 2, the following lemma is obvious.

Lemma 1. *If $\mathcal{E} \subseteq \mathcal{M}$, then $\sigma(\mathcal{E}) \subseteq \mathcal{M}$.*

In the following, we show that the family of finite disjoint union of an elementary family forms an algebra. As discussed later, a σ-algebra and an algebra are the domains of a measure and a premeasure, respectively.

Definition 3. *A non-empty family $\mathcal{E} \subseteq \mathcal{P}(X)$ is said to be an elementary family of X iif*

1) *$\phi \in \mathcal{E}$;*
2) *if E, $F \in \mathcal{E}$, then $E \cap F \in \mathcal{E}$;*
3) *if $E \in \mathcal{E}$, then E^c is a finite disjoint union of members of $\mathcal{E}$.*

Note that by the third property, X is a finite disjoint union of members of an elementary family of X.

Proposition 3. *If $\mathcal{E}$ is an elementary family of X, then the family $\mathcal{A}$ of all finite disjoint unions of members of $\mathcal{E}$ is an algebra on X.*

Proof. Let $A_1, \cdots, A_n \in \mathcal{E}$ and $A_n^c = \bigcup_{k=1}^m B_k$ for some disjoint $B_k \in \mathcal{E}$. Then, for $j \in \{1, 2, \cdots, n\}$,

$$A_j \setminus A_n = A_j \cap A_n^c = \bigcup_{k=1}^{m} (A_j \cap B_k) \in \mathcal{A} \quad (\because A_j \cap B_k \in \mathcal{E} \text{ and } \{A_j \cap B_k\}_{k=1}^m \text{ is disjoint.}).$$

This implies $\bigcup_{j=1}^n A_j = A_n \cup \left(\bigcup_{j=1}^{n-1} (A_j \setminus A_n) \right) \in \mathcal{A}$. To show that $\mathcal{A}$ is closed under complements, let $A_j^c = \bigcup_{k=1}^{m_j} B_{j,k}$ for $j \in \{1, 2, \cdots, n\}$, where $B_{j,1}, B_{j,2}, \cdots, B_{j,m_j}$ are disjoint members of $\mathcal{E}$. Then,

$$\left(\bigcup_{j=1}^{n} A_j \right)^c = \bigcap_{j=1}^{n} \left(\bigcup_{k=1}^{m_j} B_{j,k} \right) = \bigcup \big\{ \underbrace{B_{1,k_1} \cap \cdots \cap B_{n,k_n}}_{\in \mathcal{E}} : 1 \le k_j \le m_j,\ 1 \le j \le n \big\} \in \mathcal{A}$$

by the set operations over a finite number of sets, which completes the proof. □

II. Abstract Measurable Functions

Lemma 2. *The inverse mapping $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ defined by $f^{-1}(E) \doteq \{x \in X : f(x) \in E\}$ preserves the unions, intersections, and complements. That is, for any indexed family $\{E_\alpha\}_{\alpha \in A} \subseteq \mathcal{P}(Y)$ and any $E \in \mathcal{P}(Y)$,*

1) $f^{-1}\left(\bigcup_{\alpha \in A} E_\alpha \right) = \bigcup_{\alpha \in A} f^{-1}(E_\alpha)$,

2) $f^{-1}\left(\bigcap_{\alpha \in A} E_\alpha \right) = \bigcap_{\alpha \in A} f^{-1}(E_\alpha)$,

3) $f^{-1}(E^c) = (f^{-1}(E))^c$.

Proposition 4. *Any $\mathcal{M} \subseteq \mathcal{P}(X)$ given by $\mathcal{M} = \{f^{-1}(E) : E \in \mathcal{N}\}$ is a σ-algebra on X.*

Proof. Let $\{E_j\}_{j \in \mathbb{N}} \subseteq \mathcal{N}$ and $E \in \mathcal{N}$. Then, $\bigcup_{j \in \mathbb{N}} E_j \in \mathcal{N}$ and $E^c \in \mathcal{N}$ since $\mathcal{N}$ is a σ-algebra. Hence, by Lemma 2,

$$\bigcup_{j \in \mathbb{N}} f^{-1}(E_j) = f^{-1}\left(\bigcup_{j \in \mathbb{N}} E_j \right) \in \mathcal{M},$$

$$(f^{-1}(E))^c = f^{-1}(E^c) \in \mathcal{M}.$$

□

Definition 4. *A function $f : X \to Y$ is said to be $(\mathcal{M}, \mathcal{N})$-measurable, or just measurable when $\mathcal{M}$ and $\mathcal{N}$ are understood, iif:*

$$E \in \mathcal{N} \implies f^{-1}(E) \in \mathcal{M}.$$

Proposition 5. *A function $f : X \to Y$ is always $(\mathcal{M}, \mathcal{N})$-measurable for $\mathcal{M}$ given in Proposition 4,*

Proof. Trivial by Definition 4. □

The $(\mathcal{M}, \mathcal{N})$-measurability can be understood in a sense that the restricted function $f^{-1}|_{\mathcal{N}} : \mathcal{N} \to \mathcal{M}$ on the σ-algebra domain $\mathcal{N} \subseteq \mathcal{P}(Y)$ is well-defined in such a way that its image is contained by its codomain $\mathcal{M}$ as in the usual definition of a function. This observation can be summarized as follows.

Proposition 6. *$f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable iif $\operatorname{Im}(f^{-1}|_{\mathcal{N}}) \subseteq \mathcal{M}$.*

Corollary 1. *If $\overline{\mathcal{M}}$ and $\underline{\mathcal{N}}$ are respective σ-algebras on X and Y such that $\mathcal{M} \subseteq \overline{\mathcal{M}}$ and $\underline{\mathcal{N}} \subseteq \mathcal{N}$, then*

$$f \text{ is } (\mathcal{M}, \mathcal{N})\text{-measurable} \implies f \text{ is } (\overline{\mathcal{M}}, \underline{\mathcal{N}})\text{-measurable}.$$

Proof. Trivial by Proposition 6 and $\operatorname{Im}(f^{-1}|_{\underline{\mathcal{N}}}) \subseteq \operatorname{Im}(f^{-1}|_{\mathcal{N}}) \subseteq \mathcal{M} \subseteq \overline{\mathcal{M}}$. □

Corollary 2. *For any σ-algebra $\mathcal{N}$ on Y, any function $f : X \to Y$ is $(\mathcal{P}(X), \mathcal{N})$-measurable.*

Proof. Trivial by $\operatorname{Im}(f^{-1}|_{\mathcal{N}}) \subseteq \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the largest σ-algebra on X (see Example 1). □

The following is the dual to Propositions 4 and 5.

Lemma 3. *Any $\mathcal{N} \subseteq \mathcal{P}(Y)$ given by $\mathcal{N} = \{E : f^{-1}(E) \in \mathcal{M}\}$ is a σ-algebra on Y; for such $\mathcal{N}$, f is always $(\mathcal{M}, \mathcal{N})$-measurable.*

Proof. Let $\{E_j\}_{j\in\mathbb{N}} \subseteq \mathcal{N}$ and $E \in \mathcal{N}$. Then, we have $\{f^{-1}(E_j)\}_{j\in\mathbb{N}} \subseteq \mathcal{M}$ and $f^{-1}(E) \in \mathcal{M}$. Since $\mathcal{M}$ is a σ-algebra, we therefore obtain by Lemma 2

$$f^{-1}\left(\bigcup_{j\in\mathbb{N}} E_j\right) = \bigcup_{j\in\mathbb{N}} f^{-1}(E_j) \in \mathcal{M},$$

$$f^{-1}(E^c) = (f^{-1}(E))^c \in \mathcal{M},$$

meaning that $\bigcup_{j\in\mathbb{N}} E_j \in \mathcal{N}$ and $E^c \in \mathcal{N}$. Therefore, $\mathcal{N}$ is a σ-algebra on Y. Moreover, the definition of $\mathcal{N}$ gives $\operatorname{Im}(f^{-1}|_{\mathcal{N}}) \subseteq \mathcal{M}$, and hence f is $(\mathcal{M}, \mathcal{N})$-measurable by Proposition 6. □

Proposition 7. *For any subset $\mathcal{E} \subseteq Y$, f is $(\mathcal{M}, \sigma(\mathcal{E}))$-measurable iif $\operatorname{Im}(f^{-1}|_{\mathcal{E}}) \subseteq \mathcal{M}$, i.e., iif*

$$f^{-1}(E) \in \mathcal{M} \text{ for all } E \in \mathcal{E}.$$

Proof.

($\Longrightarrow$) If f is $(\mathcal{M}, \sigma(\mathcal{E}))$-measurable, then $\mathcal{E} \subseteq \sigma(\mathcal{E})$ and Proposition 6 imply that

$$\mathrm{Im}(f^{-1}|_{\mathcal{E}}) \subseteq \mathrm{Im}(f^{-1}|_{\sigma(\mathcal{E})}) \subseteq \mathcal{M}.$$

($\Longleftarrow$) The assumption assures the expression $\mathcal{E} = \{E \in \mathcal{E} : f^{-1}(E) \in \mathcal{M}\}$. Moreover, the σ-algebra $\mathcal{N}$ given in Lemma 3 obviously satisfies $\mathcal{E} = \{E \in \mathcal{E} : f^{-1}(E) \in \mathcal{M}\} \subseteq \{E \in \mathcal{P}(Y) : f^{-1}(E) \in \mathcal{M}\} = \mathcal{N}$ and hence, $\sigma(\mathcal{E}) \subseteq \mathcal{N}$ by Lemma 1. This obviously results in $\mathrm{Im}(f^{-1}|_{\sigma(\mathcal{E})}) \subseteq \mathrm{Im}(f^{-1}|_{\mathcal{N}}) \subseteq \mathcal{M}$, and hence f is $(\mathcal{M}, \sigma(\mathcal{E}))$-measurable by Proposition 6. □

Proposition 8. *Suppose $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable and $g : Y \to Z$ is $(\mathcal{N}, \mathcal{O})$-measurable. Then, the composition $g \circ f : X \to Z$ is $(\mathcal{M}, \mathcal{O})$-measurable.*

Proof. By measurability, $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$ and $g^{-1}(F) \in \mathcal{N}$ for all $F \in \mathcal{O}$. Hence, it is obvious that $f^{-1}(g^{-1}(F)) \in \mathcal{M}$ for all $F \in \mathcal{O}$, and the proof is completed by $(g \circ f)^{-1} = f^{-1} \circ g^{-1} = f^{-1}(g^{-1}(\cdot))$. □

III. Product σ-Algebra and Measurable Functions on a Cartesian Product

Let $\{Y_\alpha\}_{\alpha \in A}$ be a family of non-empty sets Y_α indexed by $\alpha \in A$.

Definition 5. *The Cartesian product $\prod_{\alpha \in A} Y_\alpha$ is the set of all functions $y : A \to \bigcup_{\alpha \in A} X_\alpha$ such that $y(\alpha) \in Y_\alpha$ for every $\alpha \in A$. We denote $y_\alpha \doteq y(\alpha)$ and call it the α-th element of y.*

If the sets Y_α are all equal to some fixed set Z, then we denote $\prod_{\alpha \in A} Y_\alpha$ by Z^A. Moreover, if

$$A \text{ is finite and given by } A = \{1, 2, 3, \cdots, n\} \text{ for some } n \in \mathbb{N}, \tag{1}$$

then $\prod_{\alpha \in A} Y_\alpha$ and Z^A will be denoted by $Y_1 \times Y_2 \times \cdots \times Y_n$ and Z^n, respectively. For notational convenience, we also denote $Y \doteq \prod_{\alpha \in A} Y_\alpha$ throughout this section.

Definition 6. *The α-th projection map $p_\alpha : Y \to Y_\alpha$ is such that $p_\alpha(y) = y_\alpha$ for all $y \in Y$.*

Remark 2. *When* (1) *is true for the indexed set A, the Cartesian product $Y = Y_1 \times Y_2 \times \cdots \times Y_n$ is usually defined as the set of all tuples $(y_1, y_2, \cdots, y_n)$ such that $y_\alpha \in Y_\alpha$ for $\alpha = 1, 2, \cdots, n$. This definition is set-theoretically different from the Cartesian product defined above in Definition 5. However, both definitions with the notation $y_\alpha \doteq y(\alpha)$ above agree that*

$$y_\alpha \in Y_\alpha \textit{ for any } \alpha \in A \textit{ and thus } p_\alpha(y) = y_\alpha \textit{ for any } \alpha \in A \textit{ and all } y \in Y. \tag{2}$$

Likewise, all of the theorems and proofs will be also true for the finite case (1).

Now, let for each $\alpha \in A$: $\begin{cases} \mathcal{N}_\alpha \subseteq \mathcal{P}(Y_\alpha) \text{ be a } \sigma\text{-algebra on } Y_\alpha; \\ f_\alpha : X \to Y_\alpha \text{ be a function from } X \text{ to } Y_\alpha \text{ indexed by } \alpha \in A. \end{cases}$

The σ-algebra $\mathcal{M}(f.) \subseteq \mathcal{P}(X)$ given by $\mathcal{M}(f.) = \sigma\big(\{f_\alpha^{-1}(E) : E \in \mathcal{N}_\alpha \text{ and } \alpha \in A\}\big)$ is called the σ-algebra (on X) generated by $\{f_\alpha\}_{\alpha\in A}$. A *product σ-algebra* on Y is the σ-algebra $\mathcal{M}(p.)$ generated by $\{p_\alpha\}_{\alpha\in A}$ when $X = Y$. We denote this σ-algebra $\mathcal{M}(p.)$ by $\bigotimes_{\alpha\in A} \mathcal{N}_\alpha$.

Proposition 9. *f_α is $(\mathcal{M}(f.), \mathcal{N}_\alpha)$-measurable for each $\alpha \in A$.*

Proof. Let $\mathcal{M}_\alpha \doteq \sigma(\{f_\alpha^{-1}(E) : E \in \mathcal{N}_a\})$. Then, Propositions 4 and 5 imply that $\mathcal{M}_\alpha$ is a σ-algebra and f_α is $(\mathcal{M}_\alpha, \mathcal{N}_\alpha)$-measurable. Since $\mathcal{M}_\alpha \subseteq \mathcal{M}(f.)$, the proof is completed by Corollary 1. $\square$

Corollary 3. *p_α is $(\bigotimes_{\alpha\in A} \mathcal{N}_\alpha, \mathcal{N}_\alpha)$-measurable for each $\alpha \in A$.*

Remark 3. *Note that $\bigcap_{\alpha\in A} \mathcal{M}_\alpha$ is also a σ-algebra (on X) by Proposition 2, and $\bigcap_{\alpha\in A} \mathcal{M}_\alpha \subseteq \mathcal{M}(f.)$ holds by $\mathcal{M}_\alpha \subseteq \mathcal{M}(f.)$, where $\mathcal{M}_\alpha \doteq \sigma(\{f_\alpha^{-1}(E) : E \in \mathcal{N}_a\})$. However, $\bigcap_{\alpha\in A} \mathcal{M}_\alpha = \mathcal{M}(f.)$ does not hold in general, and f_α is not necessarily $(\bigcap_{\alpha\in A} \mathcal{M}_\alpha, \mathcal{N}_\alpha)$-measurable.*

Proposition 10. *Suppose $\mathcal{E}_\alpha \subseteq \mathcal{P}(Y_\alpha)$ and $\mathcal{N}_\alpha = \sigma(\mathcal{E}_\alpha)$ for each $\alpha \in A$. Then,*

$$\bigotimes_{\alpha\in A} \mathcal{N}_\alpha = \sigma\Big(\Big\{p_\alpha^{-1}(E) : E \in \mathcal{E}_\alpha \text{ and } \alpha \in A\Big\}\Big). \tag{3}$$

Moreover, if $Y_\alpha \in \mathcal{E}_\alpha$ for each $\alpha \in A$, then $\bigotimes_{\alpha\in A} \mathcal{N}_\alpha \subseteq \sigma\big(\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha\}\big)$. In addition to that, if A is countable, then $\bigotimes_{\alpha\in A} \mathcal{N}_\alpha = \sigma\big(\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha\}\big)$.

Proof. Let $\mathcal{F} \doteq \{p_\alpha^{-1}(E) : E \in \mathcal{E}_\alpha \text{ and } \alpha \in A\} \subseteq \mathcal{P}(Y)$. Then, obviously, $\mathcal{F} \subseteq \bigotimes_{\alpha\in A} \mathcal{N}_\alpha$, and thus we obtain $\sigma(\mathcal{F}) \subseteq \bigotimes_{\alpha\in A} \mathcal{N}_\alpha$ by Lemma 1. On the other hand, Lemma 3 implies that for each $\alpha \in A$, $\{E : p_\alpha^{-1}(E) \in \sigma(\mathcal{F})\} \subseteq \mathcal{P}(Y_\alpha)$ is a σ-algebra on Y_α that obviously contains $\mathcal{E}_\alpha$ and thus $\mathcal{N}_\alpha$ ($\because$ it is a σ-algebra; see also the definition of $\mathcal{F}$ and $\mathcal{N}_\alpha = \sigma(\mathcal{E}_\alpha)$). In other words, for each $\alpha \in A$, we have $p_\alpha^{-1}(E) \in \sigma(\mathcal{F})$ whenever $E \in \mathcal{N}_\alpha$, meaning that $\bigotimes_{\alpha\in A} \mathcal{N}_\alpha \subseteq \sigma(\mathcal{F})$ by Lemma 1. Therefore, $\bigotimes_{\alpha\in A} \mathcal{N}_\alpha = \sigma(\mathcal{F})$.

Next, let $\mathcal{G} \doteq \{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha\}$ and $E \in \mathcal{E}_\alpha$. Then, since $p_\alpha^{-1}(E) = \prod_{\beta\in A} E_\beta$ where $E_\beta = Y_\beta \in \mathcal{E}_\beta$ for $\beta \neq \alpha$ and $E_\alpha = E \in \mathcal{E}_\alpha$, we have $\{p_\alpha^{-1}(E) : E \in \mathcal{E}_\alpha \text{ and } \alpha \in A\} \subseteq \mathcal{G} \subseteq \sigma(\mathcal{G})$ and thus $\bigotimes_{\alpha\in A} \mathcal{N}_\alpha \subseteq \sigma(\mathcal{G})$ by (3) and Lemma 1. On the other hand, let $E_\alpha \in \mathcal{E}_\alpha$ for each $\alpha \in A$ and suppose A is countable. Then, since any σ-algebra is closed under finite intersections (see Proposition 1), we have by (3) that $\bigcap_{\alpha\in A} p_\alpha^{-1}(E_\alpha) \in \bigotimes_{\alpha\in A} \mathcal{N}_\alpha$, where $\bigcap_{\alpha\in A} p_\alpha^{-1}(E_\alpha) = \prod_{\alpha\in A} E_\alpha$ holds and thus

$$\Big\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha\Big\} \subseteq \bigotimes_{\alpha\in A} \mathcal{N}_\alpha.$$

Finally, by Lemma 1, $\sigma(\mathcal{G}) \subseteq \bigotimes_{\alpha\in A} \mathcal{N}_\alpha$ and therefore $\sigma(\mathcal{G}) = \bigotimes_{\alpha\in A} \mathcal{N}_\alpha$ when A is countable. $\square$

By letting $\mathcal{E}_\alpha = \mathcal{N}_\alpha$ for all $\alpha \in A$ in Proposition 10, we obtain the following corollary.

Corollary 4. $\bigotimes_{\alpha\in A}\mathcal{N}_\alpha \subseteq \sigma\big(\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{N}_\alpha\}\big)$. *Moreover, if* A *is countable, then*

$$\bigotimes_{\alpha\in A}\mathcal{N}_\alpha = \sigma\bigg(\bigg\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{N}_\alpha\bigg\}\bigg).$$

Proof. For the completeness, we provide the independent proof, which is almost same to the second paragraph of the proof of Proposition 10. Let $\mathcal{N} \doteq \sigma\big(\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{N}_\alpha\}\big)$ and $E \in \mathcal{N}_\alpha$. Then, since $p_\alpha^{-1}(E) = \prod_{\beta\in A} E_\beta$ where $E_\beta = Y_\beta \in \mathcal{N}_\beta$ for $\beta \neq \alpha$ and $E_\alpha = E \in \mathcal{N}_\alpha$, we have $\{p_\alpha^{-1}(E) : E \in \mathcal{N}_\alpha \text{ and } \alpha \in A\} \subseteq \mathcal{N}$ and thus $\bigotimes_{\alpha\in A}\mathcal{N}_\alpha \subseteq \mathcal{N}$ by Lemma 1. On the other hand, let $E_\alpha \in \mathcal{N}_\alpha$ and suppose A is countable. Then, since any σ-algebra is closed under finite intersections (see Proposition 1), $\bigcap_{\alpha\in A} p_\alpha^{-1}(E_\alpha) \in \bigotimes_{\alpha\in A}\mathcal{N}_\alpha$, where $\bigcap_{\alpha\in A} p_\alpha^{-1}(E_\alpha) = \prod_{\alpha\in A} E_\alpha$ holds and thus $\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{N}_\alpha\} \subseteq \bigotimes_{\alpha\in A}\mathcal{N}_\alpha$. Finally, by Lemma 1, we obtain $\mathcal{N} \subseteq \bigotimes_{\alpha\in A}\mathcal{N}_\alpha$ and therefore $\mathcal{N} = \bigotimes_{\alpha\in A}\mathcal{N}_\alpha$ when A is countable. □

Proposition 11. *Let* $(X, \mathcal{M})$ *and* $(Y_\alpha, \mathcal{N}_\alpha)$ $(\alpha \in A)$ *be measurable spaces, and denote* $\mathcal{N} \doteq \bigotimes_{\alpha\in A}\mathcal{N}_\alpha$*. Then,* $f : X \to Y$ *is* $(\mathcal{M}, \mathcal{N})$*-measurable iif* $f_\alpha = p_\alpha \circ f$ *is* $(\mathcal{M}, \mathcal{N}_\alpha)$*-measurable for all* $\alpha \in A$*.*

Proof. Since every p_α is $(\mathcal{N}, \mathcal{N}_\alpha)$-measurable by Corollary 3 and the composition of measurable functions is also measurable by Proposition 8, $f_\alpha = p_\alpha \circ f : X \to Y_\alpha$ is $(\mathcal{M}, \mathcal{N}_\alpha)$-measurable for each $\alpha \in A$ if f is $(\mathcal{M}, \mathcal{N})$-measurable. Conversely, if each f_α is $(\mathcal{M}, \mathcal{N}_\alpha)$-measurable, then for all $E \in \mathcal{N}_\alpha$ and each $\alpha \in A$, $f^{-1}(p_\alpha^{-1}(E)) = f_\alpha^{-1}(E) \in \mathcal{M}$; the proof is completed by $\mathcal{N} = \sigma\big(\{p_\alpha^{-1}(E) : E \in \mathcal{N}_\alpha \text{ and } \alpha \in A\}\big)$ and Proposition 7. □

IV. Borel σ-Algebra, Some Topology, and $(\mathcal{B}_X, \mathcal{B}_Y)$-Measurability

The concept of the abstract Borel σ-algebra is essentially connected to the open sets and general topology. Thus, we begin this section with the general definition of topology and open/closed sets shown below.

Definition 7. *A topological space is an ordered pair* (X, τ_X)*, where* X *is a set and* τ_X *is a collection of subsets of* X*, satisfying the following axioms:*

1) $\phi,\ X \in \tau_X$,

2) $\{E_\alpha\}_{\alpha\in A} \subseteq \tau_X \implies \bigcup_{\alpha\in A} E_\alpha \in \tau_X$ *(closed under arbitrary unions),*

3) $E_1, E_2, \cdots, E_n \in \tau_X \implies E_1 \cap E_2 \cap \cdots \cap E_n \in \tau_X$ *(closed under finite intersections).*

τ_X *is called a topology on* X*; each member* U *of* τ_X *is said to be an open set; each complement* U^c *is called a closed set.*

If the topology τ_X is well-understood, we simply say that X is a topological space. In this section, we suppose X and Y are topological spaces and denote their topologies by τ_X and τ_Y, respectively. The open sets in τ_X and τ_Y are denoted by U and V, respectively.

Definition 8. *A Borel σ-algebra $\mathcal{B}_X$ on a topological space X is the σ-algebra generated by its topology τ_X (the collection of all open sets), i.e., $\mathcal{B}_X \doteq \sigma\{\tau_X\}$.*

A countable intersection of open sets is called a G_δ set; a countable union of closed sets is called an F_σ set; a countable union of G_δ sets is called a $G_{\delta\sigma}$ set; a countable intersection of F_σ set is called an $F_{\sigma\delta}$ set (here, δ and σ stand for intersection and union, respectively). By Proposition 1 and the definitions of a topology and a σ-algebra above, we obtain the following.

Proposition 12. *$\mathcal{B}_X$ contains the followings.*

1) all of the open sets and closed sets;

2) arbitrary unions of open sets and arbitrary intersections of closed sets;

3) all of G_δ and F_σ sets;

4) all of $G_{\delta\sigma}$ and $F_{\sigma\delta}$ sets.

Proof. Since $\tau_X \subseteq \sigma\{\tau_X\}$, $\mathcal{B}_X$ contains all of the open sets U and their arbitrary unions. Since a σ-algebra is closed under taking complements, $\mathcal{B}_X$ also contains all of the closed sets U^c and by De Morgan's laws, their arbitrary intersections. Moreover, a σ-algebra is closed under countable unions and intersections, and thereby $\mathcal{B}_X$ includes any F_σ set (any countable union of closed sets) and G_δ set (any countable intersection of open sets). Likewise, any $F_{\sigma\delta}$ sets and $G_{\delta\sigma}$ sets also belong to $\mathcal{B}_X$. $\square$

Such characterization of a σ-algebra via topology gives the following property.

Proposition 13. *Every continuous function $f : X \to Y$ is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable.*

Proof. f is continuous iif $f^{-1}(V)$ is open in X for every open set $V \in \tau_Y$. Hence, for every $V \in \tau_Y$, $f^{-1}(V) \in \mathcal{B}_X$ and the proof is complete by Propositoin 7. $\square$

Similarly to Proposition 2, we obtain the following proposition for a family of topologies.

Proposition 14. *If $\{\tau_{X,\alpha}\}_{\alpha\in A}$ is a family of topologies on X, $\tau_X = \bigcap_{\alpha\in A} \tau_{X,\alpha}$ is also a topology on X.*

Proof. Since ϕ, $X \in \tau_{X,\alpha}$ for all $\alpha \in A$, they also belongs to τ_X. Next, suppose $\{U_\beta\}_{\beta\in B} \subseteq \tau_X$, where B is an index set. Then, $U_\beta \in \tau_X$ implies that $U_\beta \in \tau_{\alpha,X}$ for all $\beta \in B$. Since $\tau_{X,\alpha}$ is a topology, we obtain $\bigcup_{\beta\in B} U_\beta \in \tau_{X,\alpha}$ for all $\alpha \in A$, implying that $\bigcup_{\beta\in B} U_\beta \in \tau_X$. Likewise, one can also show that τ_X is closed under finite intersections. Therefore, τ_X is a topology. $\square$

Definition 9. *For any family $\mathcal{U} \subseteq \mathcal{P}(X)$, $\tau(\mathcal{U})$ denotes the smallest topology on X that contains $\mathcal{U}$; we call $\tau(\mathcal{U})$ the topology generated by $\mathcal{U}$.*

Remark 4. *Note that for any family $\mathcal{U} \subseteq \mathcal{P}(X)$, there is at least one topology, namely, the power set $\mathcal{P}(X)$ itself, the largest topology on X, that contains $\mathcal{U}$. Moreover, since any (uncountable) intersection of topologies is also a topology as shown in Proposition 14, the smallest topology $\tau(\mathcal{U})$ in Definition 9 can be recognized as the intersection of all topologies containing $\mathcal{U}$. Moreover, if $\mathcal{U}$ is understood, one can construct such a topology τ_X by the following procedure:*

1) *add ϕ, every member of U, and X to τ_X;*
2) *add all finite intersections of the sets in τ_X to τ_X;*
3) *add all arbitrary unions of the sets in τ_X to τ_X.*

Here, the order of 1)–3) is strict and not interchangeable.

Proposition 15. *Let (Y, d) be a metric space and $B(y, r) \doteq \{z \in Y : d(y, z) < r\}$ be an open ball centerred at $y \in Y$ with its radius $r > 0$. Define $\tau_d \subseteq \mathcal{P}(Y)$ as*

$$V \in \tau_d \text{ iif } \forall y \in V : \; \exists r > 0 \text{ such that } B(y, r) \subseteq V. \tag{4}$$

Then, τ_d is a topology on Y. Moreover, every $V \in \tau_d$ is a union of open balls.

Proof. First, note that $Y \in \tau_d$ is true since $B(y, r) \subseteq Y$ holds for any $r > 0$ by the definition of $B(y, r)$. Moreover, (4) is true vacuously (any statement that starts with $\forall x \in \phi$ is true, as there is no x in the empty set to falsify the rest of the statement). Hence, $\phi \in \tau_d$. Next, let $\{V_\alpha\}_{\alpha \in A} \subseteq \tau_d$ and $y \in V_\beta$ for some $\beta \in A$. Then, there exists $r > 0$ such that $B(y, r) \subseteq V_\beta$ and hence, $B(y, r) \subseteq \bigcup_{\alpha \in A} V_\alpha$, that is, $\bigcup_{\alpha \in A} V_\alpha \in \tau_d$. In a similar manner, suppose V_1, V_2, $\cdots$, $V_n \in \tau_d$ and $y \in V_j$ for all $j = 1, 2, \cdots, n$. Then, for each j, there is $r_j > 0$ such that $B(y, r_j) \subseteq V_j$. Hence, $B(y, r) \subseteq V_j$ for all $j = 1, 2, \cdots, n$, where $r = \min\{r_j\}_{j=1}^n$. That is, $B(y, r) \subseteq \bigcap_{j=1}^n V_j$ and thereby $\bigcap_{j=1}^n V_j \in \tau_d$. Therefore, τ_d is a topology on Y.

Moreover, for any $V \in \tau_d$, it is obvious that $\bigcup_{y \in V} B(y, r_y) \subseteq V$ for some $r_y > 0$ by (4). Conversely, since each $y \in V$ is obviously contained in $B(y, r_y)$, $V = \{y \in V\} \subseteq \bigcup_{y \in V} B(y, r_y)$. Therefore, every $V \in \tau_d$ is the union of open balls $B(y, r_y)$. $\square$

The topology τ_d in Proposition 15 is called the topology on Y generated by the metric d. Since every $V \in \tau_d$ is a union of open balls by Proposition 15, the collection of all open balls

$$\mathcal{E} = \big\{B(y, r) : y \in Y \text{ and } r > 0\big\} \tag{5}$$

is a base of the topology τ_d.

Definition 10. *A collection $\mathcal{E}$ of subsets of Y is said to be a base of a topology τ_Y on Y iif every $V \in \tau_Y$ is a union of members of $\mathcal{E}$. The topological space Y is said to be second countable iif its topology τ_Y has a countable base $\mathcal{E}$.*

Definition 11. *A subset D of Y is said to be dense iif every point $y \in Y$ either belongs to D or is a limit point of D. A topological space Y is separable if it contains a countable, dense subset D.*

Proposition 16. *A metric space is separable iif it is second countable.*

Proof. Every second countable space is also separable. To prove the necessity, suppose Y is a separable metric space. Then, Y contains a countable, dense subset D. The set $\mathbb{Q}$ of all rational numbers is countable, so

$$\mathfrak{B} \doteq \{B(y,r) : y \in D \text{ and } 0 < r \in \mathbb{Q}\}$$

is a countable collection of open balls. Let τ_d be the topology on Y generated by d and $V \in \tau_d$. Then, for each $\bar{y} \in V$, there is $\bar{r} > 0$ such that $B(\bar{y}, \bar{r}) \subseteq V$. Next, if $\bar{y} \in V \cap D$, set $y = \bar{y}$; otherwise, choose $y \in D$ such that $d(\bar{y}, y) < r$ for a rational number $r \in (0, \bar{r}/2]$. Then, by triangular inequality, for any $z \in B(y,r)$, we obtain $d(\bar{y}, z) \le d(\bar{y}, y) + d(y,z) < 2r \le \bar{r}$, which implies $z \in B(\bar{y}, \bar{r})$. In summary,

$$B(y,r) \subseteq B(\bar{y}, \bar{r}) \subseteq V \text{ and } \bar{y} \in B(y,r).$$

Therefore, for each $\bar{y} \in V$, there is $y \in D$ and a rational number $r_y > 0$ such that $\bar{y} \in B(y, r_y) \subseteq V$, where $B(y, r_y) \in \mathfrak{B}$. This implies

$$\bigcup_{y \in D} B(y, r_y) \subseteq V = \{\bar{y} \in V\} \subseteq \bigcup_{y \in D} B(y, r_y).$$

Therefore, $V = \bigcup_{y \in D} B(y, r_y)$, that is, Y is second countable. □

Now, let Y_j $(j = 1, 2, \cdots, n)$ be a metric space with its metric $d_j : Y_j \times Y_j \to \mathbb{R}_+$ and

$$Y \doteq \prod_{j=1}^{n} Y_j = Y_1 \times Y_2 \times \cdots \times Y_n.$$

Then, Y is a metric space with the product metric $d : Y \times Y \to \mathbb{R}_+$:

$$d(y,z) \doteq \max\{d_1(y_1, z_1), d_2(y_2, z_2), \cdots, d_n(y_n, z_n)\}, \tag{6}$$

where $y = (y_1, y_2, \cdots, y_n) \in Y$ and $z = (z_1, z_2, \cdots, z_n) \in Y$. Let $\mathcal{B}_{Y_j} = \sigma\{\tau_{d_j}\}$ $(j = 1, 2, \cdots, n)$ and $\mathcal{B}_Y = \sigma\{\tau_d\}$ be the Borel σ-algebras generated by the topologies τ_{d_j} and τ_d on Y_j and Y generated by d_j and d, respectively. Denote the open balls in Y and Y_j $(j = 1, 2, \cdots, n)$ by $B(y,r)$ and $B_j(y_j, r)$ for $y \in Y$ and $y_j \in Y_j$, respectively.

Lemma 4. *$B(y,r)=\prod_{j=1}^{n} B_j(y_j,r)$ for each $y=(y_1,y_2,\cdots,y_n)\in Y$ and $r>0$.*

Proof. Every $z=(z_1,z_2,\cdots,z_n)\in B(y,r)$ satisfies $d(y,z)<r$ which holds iif $d(y_j,z_j)<r$ for all $j=1,2,\cdots,n$ by the definition of the product metric (6). This directly proves $B(y,r)=\prod_{j=1}^{n} B_j(y_j,r)$. □

Lemma 5. *If Y_j's are all separable, then so is Y.*[1]

Proof. For each $i=1,2,\cdots,n$, let $D_j\subseteq Y_j$ be a countable dense subset of Y_j and $y=(y_1,y_2,\cdots,y_n)\in Y$ with each $y_j\in Y_j$. Then, since D_j is dense in Y_j, there is a sequence $\{z_{j,k}\}_{k=1}^{\infty}$ in D_j such that

$$\lim_{k\to\infty} d_j(y_j,z_{j,k})=0 \tag{7}$$

(if $y_k\in D_j$, $\{z_{j,k}\}_{k=1}^{\infty}$ with $z_{j,k}=y_k\in D_j$ for all k is trivially such a sequence). For each $k\in\mathbb{N}$, let $z^{(k)}\doteq(z_{1,k},z_{2,k},\cdots,z_{3,k})\in D$, where $D\doteq\prod_{j=1}^{n} D_j$. Then, by (7) and the definition of the metric d,

$$\lim_{k\to\infty} d(y,z^{(k)})=\lim_{k\to\infty}\Big(\max\{d(y_1,z_{1,k}),d(y_2,z_{2,k}),\cdots,d(y_n,z_{n,k})\}\Big)=0,$$

which implies that every $y\in Y$ is a limit point of D and hence, D is a dense subset of Y. Moreover, D is countable since any finite product of countable subsets is also countable. Hence, Y is separable. □

Lemma 6. *Every projection map $p_j:Y\to Y_j$ is continuous.*

Proof. Suppose y, $z\in Y$ and let y_j, $z_j\in Y_j$ be their j-th elements. Then, we have $p_j(y)=y_j$ and $p_j(z)=z_j$. Set $\delta=\varepsilon>0$. Then, by the definition of the produce metric (6),

$$d(y,z)<\delta \implies d_j(p_j(y),p_j(z))=d_j(y_j,z_j)<\varepsilon,$$

implying continuity of $p_j:(Y,d)\to(Y_j,d_j)$. □

Proposition 17. *$\bigotimes_{j=1}^{n}\mathcal{B}_{Y_j}\subseteq\mathcal{B}_Y$. Moreover, if Y_j's are all separable, then $\bigotimes_{j=1}^{n}\mathcal{B}_{Y_j}=\mathcal{B}_Y$.*

Proof. By Proposition 10, we have $\bigotimes_{j=1}^{n}\mathcal{B}_{Y_j}=\sigma\big(\{\pi_j^{-1}(V_j):V_j\in\tau_{d_j}\text{ and }j=1,2,\cdots,n\}\big)$, where V_j is open in Y_j. Since p_j is continuous by Lemma 6, $\pi_j^{-1}(V_j)$ is open in Y. Hence, $\bigotimes_{j=1}^{n}\mathcal{B}_{Y_j}\subseteq\mathcal{B}_Y$ by Lemma 1. Moreover, suppose Y_j's are all separable. Then, they are all second countable by Proposition 16, so that there exist countable bases $\mathfrak{B}_j\subseteq\tau_{d_j}$, i.e., countable collection of open balls B_j, such that every $V_j\in Y_j$ is a union of members of $\mathfrak{B}_j$. Here, the union is actually a countable union since $\mathfrak{B}_j$ is countable, which yields $\mathcal{B}_{Y_j}=\sigma\{\mathfrak{B}_j\}$ ($j=1,2,\cdots,n$). Since Y is also second countable by Lemma 5 and Proposition 16, and its base $\mathfrak{B}$ is a *countable* collection of open balls B. Since $B=\prod_{j=1}^{n} B_j$ by Lemma 4, we therefore obtain $\mathcal{B}_Y=\sigma\big\{\prod_{j=1}^{n} B_j:B_j\in\mathfrak{B}_j\big\}$ and $\bigotimes_{j=1}^{n}\mathcal{B}_{Y_j}=\mathcal{B}_Y$ by Proposition 10. □

[1]In the most general situation, every countable product of separable topological spaces is also separable.

Since $\mathbb{R}$ is separable, we obtain the following corollary.

Corollary 5. $\mathcal{B}_{\mathbb{R}^n} = \bigotimes_1^n \mathcal{B}_{\mathbb{R}}$.