Necessity of Restricting the Domain of a Volume Function

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This note provides the detailed proof of the following theorem, which is also shown in Chapter 1.1 in Folland's real analysis book. In the case $n = 1$, the theorem justifies the fact that why the domain of the volume function μ (a.k.a. Lebesque measure) to measure the volume of a subset of $\mathbb{R}^n$ has not to be the entire family $\mathcal{P}(\mathbb{R}^n)$. In real analysis, the Lebesque measure μ is defined on a subset of $\mathcal{P}(\mathbb{R}^n)$ known as the family of Lebesque measurable sets in $\mathbb{R}^n$.

Theorem 1. *Let $\mu : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ be a function such that*

1) if E_1, E_2, $\cdots$ is a finite or infinite sequence of disjoint subsets of $\mathbb{R}^n$, then

$$\mu(E_1 \cup E_2 \cup \cdots) = \mu(E_1) + \mu(E_2) + \cdots ;$$

2) if $E \subseteq \mathbb{R}^n$ is congruent to $F \subseteq \mathbb{R}^n$ (that is, if E can be transformed into F by translations, rotations, and reflections), then $\mu(E) = \mu(F)$;

3) $\mu(Q) = 1$, where Q is the unit cube $Q = \{x \in \mathbb{R}^n : 0 \leq x_j < 1 \text{ for } j = 1, 2, \cdots, n\}$.

The above three conditions are inconsistent for $n = 1$.

Proof. Define an equivalence relation $x \sim y$ by declaring $x \sim y$ iff $x - y$ is rational. Let N be a subset of $[0, 1)$ that contains precisely one member of each equivalence class

$$[x] = \{y \in \mathbb{R} : x \sim y\} \text{ for } x \in [0, 1)$$

(To find such an N, the axiom of choice must be invoked). Next, let $R = \mathbb{Q} \cap [0, 1)$, and for each $r \in R$, let $N_r = N_r^{(1)} \cup N_r^{(2)}$, where

$$N_r^{(1)} = \underbrace{\{z + r : z \in N \cap [0, 1 - r)\}}_{[0,1-r)\to[r,1)} \subset [r, 1),$$

$$N_r^{(2)} = \underbrace{\{z + r - 1 : z \in N \cap [1 - r, 1)\}}_{[1-r,1)\to[0,r)} \subset [0, r).$$

To obtain N_r, shift N to the right by r units and then shift the part that sticks out beyond $[0, 1)$ one unit to the left. Then, we have $N_r \subset [0, 1)$ and the following claim.

Claim 1. *Every $x \in [0, 1)$ belongs to precisely one N_r.*

Proof of Claim 1. Let y be the member of N that belongs to the equivalence class $[x]$ for $x \in [0, 1)$, where $[x] \cap N \subset [0, 1)$. Note that either $y \in N \cap [0, 1 - r)$ or $y \in N \cap [1 - r, 1)$ holds for any $r \in [0, 1)$

($\because y \in N$). First, assume $x \geq y$ and take $r = x - y \geq 0$. Then, y should belong to $N \cap [0, 1-r)$, not to $N \cap [1-r, 1)$, since we have

$$1 - r = 1 - x + y > y \text{ by } x < 1.$$

So, in this case, "$y \in N \cap [0, 1-r)$" yields

$$x \in \{x\} = \{y + (x - y)\} = \{y + r\} \subset \{z + r : z \in N \cap [0, 1-r)\} = N_r^{(1)} \subseteq N_r,$$

and we have $x \in N_r$. Next, assume $x < y$ and take $r = x - y + 1 \geq 0$. Then, we have $y \in N \cap [1-r, 1)$ since

$$1 - r = 1 - (x - y + 1) = y - x < y \text{ by } x > 0.$$

So, in this latter case $x < y$, "$y \in N \cap [1-r, 1)$" yields

$$x \in \{x\} = \{y + (x - y + 1) - 1\} = \{y + r - 1\} \subset \{z + r - 1 : z \in N \cup [1-r, 1)\} = N_r^{(2)} \subset N_r.$$

Therefore, for both cases $x \geq y$ and $x < y$, we have $x \in N_r$ for some $r > 0$.

The remaining part for the proof of the claim is to show that if $x \in N_r \cap N_s$ for $r, s \in R$, then $r = s$. If $x \in N_r \cap N_s$ and $r \neq s$, then since r and s are rational, y_r and y_s defined, respectively, as

$$\begin{cases} y_r = x - r \text{ or } x - r + 1, \\ y_s = x - s \text{ or } x - s + 1 \end{cases}$$

are distinct elements of N that belong to the same equivalence class $[x]$, which is impossible. So, r and s must be equal to each other. □

Now, consider the volume function $\mu : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ that satisfies the above three properties 1) through 3). Then, by 1) and 2),

$$\begin{aligned} \mu(N) &= \mu(N \cap [0, 1-r)) + \mu(N \cap [1-r, 1)) \\ &= \mu(\underbrace{\{z + r : z \in N \cap [0, 1-r)\}}_{=N_r^{(1)}}) + \mu(\underbrace{\{z + r : z \in N \cap [1-r, 1)\}}_{=N_r^{(2)}}) \\ &= \mu(N_r^{(1)} \cup N_r^{(2)}) = \mu(N_r). \end{aligned}$$

for any $r \in R$. Since R is countable and $[0, 1)$ is the disjoint union of the N_r's (N_r, $N_s \subset [0, 1)$ for any $r, s \in R$, but N_r and N_s do not have an element in $[0, 1)$ in common unless $r = s$), we have

$$\mu([0, 1)) = \sum_{r \in R} \mu(N_r)$$

by 1). However, by iii), we have

$$1 = \mu([0,1)) = \sum_{r \in R} \mu(N_r) = \sum_{r \in R} \mu(N),$$

where the right hand side is either 0 (if $\mu(N) = 0$) or ∞ (if $\mu(N) > 0$), which is impossible. Hence, no such μ satisfying 1) through 3) exists, the completion of the proof. □

Remark 1. *N constructed in the proof contains one and only one rational number from $[0,1)$. Note that N contains exactly one member from each equivalence class under $\sim$, which means that:*

1) *for every $x \in [0,1)$, there exists $y \in N$ such that $x \sim y$, i.e., such that $x - y \in \mathbb{Q}$;*
2) *for every $x, y \in N$, if $x \sim y$, then $x = y$.*

In particular, we know that there exists $x \in N$ such that $0 \sim x$; since this requires that $-x \in \mathbb{Q}$, such x is rational. On the other hand, if $y \in N$ is rational, then $x - y \in \mathbb{Q}$, hence $x \sim y$, so by the second property above, we conclude that $x = y$. So, N contains one and only one rational from $[0,1)$.