Proofs of Some Propositions and Exercises in Chapter 1.3

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This note provides proofs of some propositions and exercises shown in "Chapter 1.3 Measures" in Folland's real analysis book.

Definition 1. *A measure μ is semifinite if for each measurable set E with $\mu(E) = \infty$, there exists a measurable set F such that $F \subset E$ and $0 < \mu(F) < \infty$.*

Definition 2. *A measure μ is σ-finite if X is represented as a countable union of (non-empty) measurable sets X_j's with finite measure, that is, $X = \bigcup_{j=1}^{\infty} X_j$ for $\mu(X_j) < \infty$.*

Proposition (pp.25). Let $(X, \mathcal{M}, \mu)$ be a measure space. If μ is σ-finite, then it is semifinite.

Proof. Assume μ is σ-finite. Then, there is $X_j \in \mathcal{M}$ such that $X = \bigcup_{j=1}^{\infty} X_j$ and $\mu(X_j) < \infty$ for all $j \in \mathbb{N}$. Assume without loss of generality that X_j's are disjoint and nonempty. Then, for any non-empty subset $F \in \mathcal{M}$, we have

$$F = F \cap X = F \cap \left(\bigcup_{j=1}^{\infty} X_j \right) = \bigcup_{j=1}^{\infty} F \cap X_j.$$

Assume $\mu(F) = \infty$. Then, since X_j's are all disjoint, we obtain

$$\infty = \mu(F) = \sum_{j=1}^{\infty} \mu(F \cap X_j). \tag{1}$$

Here, note that since $F \cap X_j \subseteq X_j$, we have $\mu(F \cap X_j) \leq \mu(X_j) < \infty \ \forall j \in \mathbb{N}$ by monotonicity. Since $\mu(F \cap X_j) \geq 0 \ \forall j \in \mathbb{N}$, (1) implies that there is $n \in \mathbb{N}$ such that $\mu(F \cap X_n) \neq 0$. So, for such $n \in \mathbb{N}$ we have $0 < \mu(F \cap X_n) < \infty$, implying that a σ-finite measure μ is semi-finite. □

Proposition (pp. 25). Let X be any nonempty set and f be any function from X to $[0, \infty]$. Define $\mu : \mathcal{P}(X) \to [0, \infty]$ as $\mu(E) = \sum_{x \in E} f(x)$ for $E \in \mathcal{P}(X)$, where $\sum_{x \in E} f(x)$ is the uncountable sum defined as

$$\sum_{x \in E} f(x) = \sup \left\{ \sum_{x \in F} f(x) : F \subseteq E, \ F \text{ is finite} \right\}.$$

Then,

1) μ is a measure on the measurable space $(X, \mathcal{P}(X))$;
2) the measure μ is semi-finite iff $f(x) < \infty$ for all $x \in X$;
3) the measure μ is σ-finite iff μ is semi-finite and $\{x : f(x) > 0\}$ is countable.

Proof. First, let A be defined by $A := \{x \in E : f(x) > 0\}$ for $E \in \mathcal{P}(X)$. If A is uncountable, we have $\sum_{x\in E} f(x) = \infty$, and if A is countable, then $\sum_{x\in E} f(x)$ becomes a usual (infinite) series (see Proposition 0.20 in Folland's real analysis book).

- $\mu(\phi) = \sum_{x\in\phi} f(x)$ is an empty sum which is considered zero (search "empty sum" in Wikipedia).
- Let $E \in \mathcal{P}(X)$ be expressed as $E = \bigcup_{j=1}^{\infty} E_j$ where $E_j \in \mathcal{P}(X)$ are disjoint subsets of $E \subseteq X$. Define A_j as $A_j := \{x \in E_j : f(x) > 0\}$.
  - If there is $j \in \mathbb{N}$ such that A_j is uncountable, then so is A. Therefore, we have $\sum_{x\in E_j} f(x) = \infty$ and $\sum_{x\in E} f(x) = \infty$, which yield
  $$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{x\in E} f(x) = \infty, \quad \sum_{j=1}^{\infty} \mu(E_j) = \sum_{j=1}^{\infty}\sum_{x\in E_j} f(x) = \infty.$$
  - Assume that A_j is countable for all $j \in \mathbb{N}$. Then, A is also countable so $\sum_{x\in E} f(x)$ is a usual infinite series in this case. Therefore, since E_j's are disjoint, we have
  $$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{x\in\bigcup_{j=1}^{\infty} E_j} f(x) = \sum_{j=1}^{\infty}\sum_{x\in E_j} f(x) = \sum_{j=1}^{\infty} \mu(E_j).$$

By the above arguments, μ is a measure on the measurable space $(X, \mathcal{P}(X))$.

Second, assume $f(x) < \infty \ \forall x \in X$ and $E \in \mathcal{P}(X)$ satisfies

$$\mu(E) = \sum_{x\in E} f(x) = \infty. \tag{2}$$

From (2) and $f(x) < \infty$, it can be seen that there is at least one $x_j \in E$ such that $0 < f(x_j) < \infty$. Then, if we choose a finite subset $F \subset E$ that contains x_j, then obviously its subset $\{x \in F : f(x) > 0\}$ is not empty. Therefore, we have $0 < \mu(F) = \sum_{x\in F} f(x) < \infty$. That is,

$$f(x) < \infty \ \ \forall x \in X \implies \mu \text{ is semi-finite.}$$

Now, assume μ is semi-finite and $f(x) = \infty$ for some $x \in X$. Then, for such x, we have $\mu(\{x\}) = \sum_{x\in\{x\}} f(x) = \infty$. However, the subsets of the singleton $\{x\}$ are just ϕ and $\{x\}$, and $\mu(\phi) = 0$ and $\mu(\{x\}) = \infty$, so there is no subset $F \subset E$ satisfying $0 < \mu(F) < \infty$ when $E = \{x\}$. That is, μ is not semi-finite, a contradiction. Therefore,

$$\mu \text{ is semi-finite} \implies f(x) < \infty \ \ \forall x \in X.$$

Lastly, let E_j be the disjoint subsets of X such that $X = \bigcup_{j=1}^{\infty} E_j$. Since a finite measure ($\mu(X) < \infty$) is always σ-finite, we assume $\mu(X) = \infty$ without loss of generality.

- Assume that μ is σ-finite. Then, it is semi-finite. Let E_j satisfy $\mu(E_j) < \infty$ for all $j \in \mathbb{N}$. Suppose $\{x \in X : f(x) > 0\}$ is uncountable. Then, since

$$\exists j \in \mathbb{N} \text{ s.t. } \{x \in E_j : f(x) > 0\} \text{ is uncountable} \iff \bigcup_{j=1}^{\infty} \{x \in E_j : f(x) > 0\} \text{ is uncountable,}$$

  and $X = \bigcup_{j=1}^{\infty} E_j$, there is $j \in \mathbb{N}$ such that $\{x \in E_j : f(x) > 0\}$ is uncountable. Therefore, by Proposition 0.20 in Folland's real analysis book, we have $\mu(E_j) = \infty$, a contradiction. So, $\{x \in X : f(x) > 0\}$ should be countable.
- Assume that μ is semi-finite and $A = \{x \in X : f(x) > 0\}$ is countable. Then, semi-finiteness implies $f(x) < \infty \ \forall x \in X$, and countability implies $\mu(X) = \sum_{x \in X} f(x) = \sum_{j=1}^{\infty} f(x_j)$, where $x_j \in A$. If we set E_j be a subset in X that satisfies $X = \bigcup_{j=1}^{\infty} E_j$ and contains precisely one member x_j of A (by invoking axiom of choice), then

$$\mu(E_j) = f(x_j) + \underbrace{\sum_{x \in E_j \setminus \{x_j\}} f(x)}_{=0 \ (\because \sum_{x \in A^c \cap E_j} f(x) = 0)} < \infty.$$

  Therefore, even when $\mu(X) = \infty$, we have $\mu(E_j) < \infty$ for all $j \in \mathbb{N}$, so μ is σ-finite, the completion of the proof.

□

Exercise 9 (pp. 27). If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F).$$

Proof. Note that

$$\mu(E) = \mu((E \cap F) \cup (E \setminus F)) = \mu(E \cap F) + \mu(E \setminus F),$$

$$\mu(F) = \mu((F \cap E) \cup (F \setminus E)) = \mu(E \cap F) + \mu(F \setminus E).$$

From Vann diagram, we can easily see that $\mu(E \cap F) + \mu(F \setminus E) + \mu(E \cap F) = \mu(E \cup F)$. Therefore, we have $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$. □

Exercise 10 (pp. 27). Given a measure space $(X, \mathcal{M}, \mu)$ and $E \in \mathcal{M}$, define $\mu_E(A) = \mu(A \cap E)$ for $A \in \mathcal{E}$. Then, μ_E is a measure.

Proof. First, $\mu_E(\phi) = \mu(\phi \cup E) = \mu(\phi) = 0$. Assume that $\{A_j\}_{j=1}^{\infty}$ is a sequence of disjoint sets in $\mathcal{M}$. Then, $A_j \cap E$ is also disjoint since $A_j \cap E \subseteq A_j$. Therefore, we have

$$\mu_E\left(\bigcup_{j=1}^{\infty} A_j\right) = \mu\left(\bigcup_{j=1}^{\infty} A_j \cap E\right) = \sum_{j=1}^{\infty} \mu(A_j \cap E) = \sum_{j=1}^{\infty} \mu_E(A_j)$$

Therefore, μ_E is also a measure on the measurable space $(X, \mathcal{M})$ and its restricted one $(E, \mathcal{M}_E)$, where

$$\mathcal{M}_E := \{M_E \in \mathcal{M} : M_E = E \cap M \text{ for some } M \in \mathcal{M}\}.$$

$\square$