The (Lesbesque-)Radon-Nikodym Theorem and Jordan Decomposition

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This note presents the (Lebesque-)Radon-Nikodym Theorem, Jordan Decomposition, and some of the math related to them. The materials either directly come from or strongly related to those in Chapters 3.1 and 3.2 in Folland's real analysis book. The full-proof of the Lebesque-Radon-Nikodym Theorem is not provided for simplicity.

Let ν be a signed measure and μ is a positive measure on a measurable space $(X, \mathcal{M})$.

Definition 1. *ν is said to be singular with respect to μ (or ν and μ are said to be mutually singular), denoted by $\nu \perp \mu$, iif X is represented as $X = E \cup F$ for the two disjoint measurable sets E, $F \in \mathcal{M}$ such that $\nu(E) = \mu(F) = 0$.*

Definition 2. *ν is said to be absolutely continuous with respect to μ, denoted by $\nu \ll \mu$, iif*

$$\mu(E) = 0 \implies \nu(E) = 0.$$

Theorem 1. *Suppose ν is finite. Then, $\nu \ll \mu$ iif for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|\nu(E)| < \varepsilon$ whenever $\mu(E) < \delta$.*

Lemma 1. *The signed measure ν given by $\nu(E) = \int_E f d\mu$ for an extended μ-integrable function is absolutely continuous with respect to μ.*

Proof. Without loss of generality, suppose that $f \in L^+$.[1] Since f is measurable, there exists a sequence $\{\phi_j \in L^+\}$ of simple functions such that ϕ_n is monotonically increasing and pointwisely converges to f, so that $\sup_j \phi_j(x) = f(x)$. Then, consider the standard representation of each ϕ_j:

$$\phi_j(x) = \sum_{k=1}^{n} a_{jk} \chi_{E_{jk}}(x),$$

[1] Otherwise, separate f as $f = f_+ + f_-$ for f_+, $f_- \in L^+$ and then proceed with f_+ and f_- separately, instead of f.

where $a_{jk} \in \mathbb{R}$ and $\{E_{jk} \in \mathcal{M}\}_{k=1}^{n}$ is a finite family of *disjoint* measurable sets E_{jk} such that $X = \bigcup_{k=1}^{n} E_{jk}$. Then, $\mu(E) = 0$ implies $\mu(E_{jk} \cap E) = 0$ for all $k \in \{1, 2, \cdots, n\}$ ($\because$ E_{jk}'s are all disjoint), and hence

$$\int_E \phi_j \, d\mu = \sum_{k=1}^{n} a_{jk} \mu(E_{jk} \cap E) = 0.$$

By MCT, we finally obtain

$$\nu(E) = \int_E f \, d\mu = \lim_{j \to \infty} \int_E \phi_j \, d\mu = 0.$$

$\square$

Corollary 1. *If $f \in L^1(\mu)$, then for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\mu(E) < \delta \implies \left| \int_E f \, d\mu \right| < \varepsilon.$$

Proof. By Theorem 1 and Lemma 1. $\square$

Theorem 2. (The Lebesque-Radon-Nikodym Theorem) *Let ν and μ are σ-finite. Then,*

1) *ν can be expressed as $\nu = \nu_1 + \nu_2$ for the two σ-finite signed measures ν_1 and ν_2 such that ν_1 is singular and ν_2 is absolutely continuous with respect to μ, i.e., $\nu_1 \perp \mu$ and $\nu_2 \ll \mu$;*
2) *there is a μ-extended integrable function $f : X \to \mathbb{R}$ such that $d\nu_2 = f d\mu$ and any two such functions are equal μ-a.e.*

Theorem 3. (Jordan Decomposition) *Let ν be the signed measure given by $\nu = f d\mu$ for some μ-extended integrable function f. Then, $\mu_+ = f_+ \, d\mu$ and $\mu_- = f_- \, d\mu$ are the unique positive measures such that $\nu = \mu_+ - \mu_-$ and $\mu_+ \perp \mu_-$, where $f_+(x) = \max\{f(x), 0\}$ and $f_-(x) = \max\{-f(x), 0\}$.*

Proof. Since $f = f_+ - f_-$, $\nu = \mu_+ - \mu_-$ is obvious. Let $E = f^{-1}([0, \infty))$ and $F = f^{-1}((-\infty, 0))$. Then, both E and F belong to $\mathcal{M}$, are disjoint (since so are $[0, \infty)$ and $(-\infty, 0)$), and $X = f^{-1}(\mathbb{R}) = E \cup F$. Moreover, $f_+(x) = 0$ for every $x \in F$ and $f_-(x) = 0$ for every $x \in E$, implying $\mu_+(F) = \mu_-(E) = 0$, that is, $\mu_+ \perp \mu_-$. $\square$

Lemma 2. *If ν is both singular and absolutely continuous with respect to μ, then $\nu = 0$.*

Proof. Suppose $X = E \cup F$ for the disjoint measurable sets E, $F \in \mathcal{M}$ such that $\nu(E) = \mu(F) = 0$. Then, $\nu(E) = 0$ implies $|\nu|(E) = 0$; $\nu \ll \mu$ implies $|\nu| \ll \mu$ and thus, $|\nu|(F) = 0$ by $\mu(F) = 0$. Therefore, we obtain $|\nu| = 0$, which again implies $\nu = 0$. $\square$

Corollary 2. **(The Radon-Nikodym Theorem)** *Let ν and μ are σ-finite and ν is absolutely continuous with respect to μ. Then, there is a μ-extended integrable function $f : X \to \mathbb{R}$ such that $d\nu = f d\mu$ and any two such functions are equal μ-a.e. Moreover, if ν is finite, then such f is μ-integrable.*

Proof. By the Lebesque-Radon-Nikodym Theorem (Theorem 2), there are two signed measures ν_1 and ν_2 such that

1) $\nu = \nu_1 + \nu_2$, $\nu_1 \perp \mu$, and $\nu_2 \ll \mu$;
2) $\nu_2 = f d\mu$ for some μ-extended integrable function f.

Moreover, $\nu \ll \mu$ by assumption, implying that $\nu(E) = 0$ whenever $\mu(E) = 0$. Since $\nu_2 \ll \mu$ also implies that $\nu_2(E) = 0$ for E satisfying $\mu(E) = 0$, we have $\nu_1(E) = 0$ whenever $\mu(E) = 0$ by $0 = \nu(E) = \nu_1(E) + \nu_2(E) = \nu_1(E)$. That is, $\nu_1 \ll \mu$. Hence, now that we have $\nu_1 \ll \mu$ and $\nu_1 \perp \mu$, Lemma 2 yields $\nu_1 = 0$ and hence, $\nu = \nu_2 = f d\mu$.

Next, suppose that ν is finite and decompose it as $\nu = \mu_+ - \mu_-$, where the positive measures $\mu_+ = f_+ \, d\mu$ and $\mu_- = f_- \, d\mu$ are the unique mutually-singular pair for ν shown in Theorem 3. Therefore,

$$|\nu|(X) = \mu_+(X) + \mu_-(X) = \int_X |f| \, d\mu < \infty,$$

where $|\nu|(X) < \infty$ was used. This proves that f is μ-integrable. □